Geometric Algebra and Star Products on the Phase Space

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Abstract

Superanalysis can be deformed with a fermionic star product into a Clifford calculus that is equivalent to geometric algebra. With this multivector formalism it is then possible to formulate Riemannian geometry and an inhomogeneous generalization of exterior calculus. Moreover it is shown here how symplectic and Poisson geometry fit in this context. The application of this formalism together with the bosonic star product formalism of deformation quantization leads then on space and space-time to a natural appearance of spin structures and on phase space to BRST structures that were found in the path integral formulation of classical mechanics. Furthermore it will be shown that Poincaré and Lie-Poisson reduction can be formulated in this formalism.

1 Introduction

Geometric algebra was initiated by early ideas of Hamilton, Grassmann and Clifford. The basic idea of geometrical algebra goes back to Clifford, who combined the scalar and the wedge product of vectors into one product in order to generalize complex analysis to spaces of arbitrary dimensions. With this geometric or Clifford product it is then possible to build up a multivector formalism that contains the structures of vector analysis, complex analysis and of spin. The description of spin was the physical motivation to resume the program of Clifford calculus after the Gibbs-Heavyside vector tuple formalism became the standard formalism in physics. This was done by Hestenes [1, 2] and independently also by Kähler [3], who generalized the Clifford structures of Dirac theory to an inhomogeneous exterior calculus and to curved spaces. Since then geometric algebra was extended into a full formalism and applied to a wide range of physical questions (see for example [4, 5]).

In [6] it was noticed that geometric algebra can be formulated in the realm of superanalysis, where a close connection to pseudoclassical mechanics appeared. In the superanalytic formulation of geometric algebra it is then possible to see that the geometric product is a fermionic star product that deforms the Grassmann structure into a Clifford structure [7]. Such a fermionic star product appeared already in the founding paper of deformation quantization [8] and was applied in [9, 10] for deformation quantization of pseudoclassical

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mechanics and the formulation of spin and Dirac theory in the star product formalism. In section two to six it will be described following [5, 11] how geometric algebra can be formulated as deformed superanalysis. The use of a fermionic star product has the advantage that geometric algebra can be unified with deformation quantization on a formal level. The structure of the resulting formalism is supersymmetric. For example one can describe rotations on the one hand with a bosonic star exponential that acts on the bosonic coefficients of a multivector and on the other hand with fermionic star exponentials, i.e. rotors, that act on the fermionic basis vectors of the multivector. A multivector is then invariant under a combination of such rotations, which leads to a new supersymmetric invariance condition. As shown in section seven the condition of such an invariance on space and space-time leads to a natural appearance of spin structures.

In section eight the superanalytic formulation of geometric algebra will be applied to describe symplectic geometry and Hamiltonian dynamics [12]. Using a superanalytic formalism leads to the question if the fermionic degrees of freedom on the phase space play a physical role just as the fermionic structures of space and space-time constitute the spin. Moreover one can wonder if there is a supersymmetry in classical mechanics. Supersymmetric structures in classical mechanics were first noticed by Gozzi et al. in the path integral description of classical mechanics [13, 14, 15]. The classical path integral is a path integral where all possible paths are constrained by a delta function to the classical path. The delta function can be written in terms of fermionic ghost degrees of freedom and the corresponding Lagrange function that leads to the reduction to the classical path as a gauge conditon was shown to be invariant under a BRST- and an anti-BRST-transformation. Furthermore it was shown that the fermionic ghosts could be interpreted as differential forms on the phase space [16]. Together with the star product formalism this was extended in [17] to a proposal for a differential calculus in quantum mechanics. It will be shown in section nine that these structures are the natural structures of geometric algebra that appear if one considers bosonic and fermionic time development on the phase space.

In the last two sections Poisson geometry and phase space reduction will be discussed. It will be shown that geometric algebra leads in this purely classical problem to a very elegant formulation. Especially for the example of the rigid body one sees that the dynamics is transferred by a fermionic rotor transformation from the vector to the bivector level, which is the same idea that is applied in the Kustanheimo-Stiefel transformation [18].

2 Geometric Algebra and the Clifford Star Product

The Grassmann calculus of superanalysis can be deformed with a fermionic star product into a Clifford calculus. This Clifford calculus is a multivector calculus that is equivalent to geometric algebra. In this superanalytic formulation of geometric algebra the supernumbers correspond to the multivectors and the fermionic star product corresponds to the geometric or Clifford product. If one considers for example a d-dimensional vector space with metric η_{ij}, the basis vectors of this vector space are the Grassmann variables $\boldsymbol{\sigma}_i$, $i = 1, \ldots, d$. A vector $\boldsymbol{v} = v^i \boldsymbol{\sigma}_i$ is then a supernumber of Grassmann grade one and a general multivector A is a supernumber

$$A = A^0 + A^i \boldsymbol{\sigma}_i + \frac{1}{2!} A^{i_1 i_2} \boldsymbol{\sigma}_{i_1} \boldsymbol{\sigma}_{i_2} + \ldots + \frac{1}{d!} A^{i_1 \ldots i_d} \boldsymbol{\sigma}_{i_1} \boldsymbol{\sigma}_{i_2} \ldots \boldsymbol{\sigma}_{i_d}. \tag{2.1}$$

The multivector A is called r-vector if the highest appearing Grassmann grade is r, i.e. if $A = \langle A \rangle_0 + \langle A \rangle_1 + \ldots + \langle A \rangle_r$, where $\langle \, \rangle_n$ projects onto the term of Grassmann grade n. A multivector $A_{(r)}$ is homogeneous or an r-blade if all appearing terms have the same grade, i.e. if $A_{(r)} = \langle A_{(r)} \rangle_r$.

On the above vector space one can then define the Clifford star product

$$A *_C B = A \exp\left[\sum_{i,j=1}^{d} \eta_{ij} \frac{\overleftarrow{\partial}}{\partial \boldsymbol{\sigma}_i} \frac{\overrightarrow{\partial}}{\partial \boldsymbol{\sigma}_j}\right] B. \tag{2.2}$$

As a star product the Clifford star product acts in a distributive and associative manner. The Clifford star product of two basis vectors is $\boldsymbol{\sigma}_i *_C \boldsymbol{\sigma}_j = \eta_{ij} + \boldsymbol{\sigma}_i \boldsymbol{\sigma}_j$ and with the metric one has further $\boldsymbol{\sigma}_i *_C \boldsymbol{\sigma}^j = \delta_i^j + \boldsymbol{\sigma}_i \boldsymbol{\sigma}^j$ and $\boldsymbol{\sigma}^i *_C \boldsymbol{\sigma}^j = \eta^{ij} + \boldsymbol{\sigma}^i \boldsymbol{\sigma}^j$. For two homogeneous multivectors $A_{(r)}$ and $B_{(s)}$ the Clifford star product is the sum

$$A_{(r)} *_C B_{(s)} = \langle A_{(r)} *_C B_{(s)} \rangle_{r+s} + \langle A_{(r)} *_C B_{(s)} \rangle_{r+s-2} + \dots + \langle A_{(r)} *_C B_{(s)} \rangle_{|r-s|}. \tag{2.3}$$

The term of lowest and highest grade correspond to the inner and the outer product

$$A_{(r)} \cdot B_{(s)} = \langle A_{(r)} *_C B_{(s)} \rangle_{|r-s|} \qquad \text{and} \qquad A_{(r)} B_{(s)} = \langle A_{(r)} *_C B_{(s)} \rangle_{r+s}. \tag{2.4}$$

Especially for the basis vectors one has $\boldsymbol{\sigma}_i \cdot \boldsymbol{\sigma}_j = \eta_{ij}$, $\boldsymbol{\sigma}_i \cdot \boldsymbol{\sigma}^j = \delta_i^j$ and $\boldsymbol{\sigma}^i \cdot \boldsymbol{\sigma}^j = \eta^{ij}$.

As a first example one can consider the two dimensional euclidian case. A general element of the Clifford algebra is a supernumber $A = A^0 + A^1 \boldsymbol{\sigma}_1 + A^2 \boldsymbol{\sigma}_2 + A^{12} \boldsymbol{\sigma}_1 \boldsymbol{\sigma}_2$ and a vector corresponds to the supernumber $\boldsymbol{a} = a^1 \boldsymbol{\sigma}_1 + a^2 \boldsymbol{\sigma}_2$. The Clifford star product of two vectors is

$$\boldsymbol{a} *_C \boldsymbol{b} = \boldsymbol{a}\boldsymbol{b} + \boldsymbol{a}\left[\sum_{n=1}^{2} \frac{\overleftarrow{\partial}}{\partial \boldsymbol{\sigma}_n} \frac{\overrightarrow{\partial}}{\partial \boldsymbol{\sigma}_n}\right] \boldsymbol{b} = (a^1 b^2 - a^2 b^1) \boldsymbol{\sigma}_1 \boldsymbol{\sigma}_2 + a^1 b^1 + a^2 b^2 \equiv \boldsymbol{a} \wedge \boldsymbol{b} + \boldsymbol{a} \cdot \boldsymbol{b}, \tag{2.5}$$

The basis bivector $I_{(2)} = \boldsymbol{\sigma}_1 \boldsymbol{\sigma}_2$ fulfills $I_{(2)} *_C I_{(2)} = I_{(2)}^{2*_C} = -1$ and $\overline{I_{(2)}} = -I_{(2)}$, where the involution reverses the order of the Grassmann elements:

$$\overline{\boldsymbol{\sigma}_{i_1} \dots \boldsymbol{\sigma}_{i_r}} = \boldsymbol{\sigma}_{i_r} \dots \boldsymbol{\sigma}_{i_1}. \tag{2.6}$$

$I_{(2)}$ plays in two dimensions the role of a imaginary unit.

In three dimensions a general Clifford number can be written as

$$A = A^0 + a^1 \boldsymbol{\sigma}_1 + A^2 \boldsymbol{\sigma}_2 + A^3 \boldsymbol{\sigma}_3 + A^{12} \boldsymbol{\sigma}_1 \boldsymbol{\sigma}_2 + A^{13} \boldsymbol{\sigma}_3 \boldsymbol{\sigma}_1 + A^{23} \boldsymbol{\sigma}_2 \boldsymbol{\sigma}_3 + A^{123} \boldsymbol{\sigma}_1 \boldsymbol{\sigma}_2 \boldsymbol{\sigma}_3 \tag{2.7}$$

and it contains a scalar, a vector, a bivector and a trivector or pseudoscalar part. The basis bivectors $\mathtt{Q}_1 = \boldsymbol{\sigma}_2 \boldsymbol{\sigma}_3$, $\mathtt{Q}_2 = \boldsymbol{\sigma}_1 \boldsymbol{\sigma}_3$ and $\mathtt{Q}_3 = \boldsymbol{\sigma}_1 \boldsymbol{\sigma}_2$ fulfill

$$\mathtt{Q}_1^{2*_C} = \mathtt{Q}_2^{2*_C} = \mathtt{Q}_3^{2*_C} = \mathtt{Q}_1 *_C \mathtt{Q}_2 *_C \mathtt{Q}_3 = -1 \tag{2.8}$$

which means that the even multivector $Q = q^0 + q^i \mathtt{Q}_i$ is a quaternion. The trivector part with basis $I_{(3)} = \boldsymbol{\sigma}_1 \boldsymbol{\sigma}_2 \boldsymbol{\sigma}_3$ can be used to describe the duality of vectors $\boldsymbol{b} = b^1 \boldsymbol{\sigma}_1 + b^2 \boldsymbol{\sigma}_2 + b^3 \boldsymbol{\sigma}_3$ and bivectors $\mathtt{B} = b^1 \boldsymbol{\sigma}_2 \boldsymbol{\sigma}_3 + b^2 \boldsymbol{\sigma}_3 \boldsymbol{\sigma}_1 + b^3 \boldsymbol{\sigma}_1 \boldsymbol{\sigma}_2$, i.e. $\mathtt{B} = I_{(3)} *_C \boldsymbol{b}$. With this relation one can then write the geometric product of two vectors $\boldsymbol{a} = a^1 \boldsymbol{\sigma}_1 + a^2 \boldsymbol{\sigma}_2 + a^3 \boldsymbol{\sigma}_3$ and $\boldsymbol{b} = b^1 \boldsymbol{\sigma}_1 + b^2 \boldsymbol{\sigma}_2 + b^3 \boldsymbol{\sigma}_3$ as:

$$\boldsymbol{a} *_C \boldsymbol{b} = \boldsymbol{a} \cdot \boldsymbol{b} + I_3 *_C (\boldsymbol{a} \times \boldsymbol{b}), \tag{2.9}$$

where $\boldsymbol{a} \cdot \boldsymbol{b} = \sum_{k=1}^{3} a^k b^k$ and $\boldsymbol{a} \times \boldsymbol{b} = \varepsilon_{kl}{}^m a^k b^l \boldsymbol{\sigma}_m$.

3 Vector Manifolds

In geometric algebra the points of a manifold are treated as vectors, so that a manifold can be seen as a surface in a flat ambient space. The at least $(d+1)$-dimensional flat ambient space is spanned by the rectangular basis vectors $\boldsymbol{\sigma}_a$ and is equipped with the constant metric η_{ab}. A d-dimensional vector manifold with coordinates x^i, $i = 1, \dots, d$ that is embedded in this ambient vector space is then described by smooth functions $f^a(x^i)$ and has the form $\boldsymbol{x}(x^i) = f^a(x^i)\boldsymbol{\sigma}_a$, one also uses the notation $\boldsymbol{x}(x^i) = x^a(x^i)\boldsymbol{\sigma}_a$. The vectors

$$\boldsymbol{\xi}_i(\boldsymbol{x}) = \frac{\partial \boldsymbol{x}}{\partial x^i} \tag{3.10}$$

are the frame vectors of the manifold, which can be expanded in the ambient space as $\boldsymbol{\xi}_i(\boldsymbol{x}) = \xi_i^a(\boldsymbol{x})\boldsymbol{\sigma}_a$. The $\boldsymbol{\xi}_i(\boldsymbol{x})$ span the tangent space $T_{\boldsymbol{x}}M$, with the Clifford star product

$$F *_C G = F \exp\left[\sum_{i,j=1}^{d} g_{ij}(\boldsymbol{x}) \frac{\overleftarrow{\partial}}{\partial \boldsymbol{\xi}_i} \frac{\overrightarrow{\partial}}{\partial \boldsymbol{\xi}_j}\right] G. \tag{3.11}$$

The scalar product of two frame vectors can be calculated internally and externally as $g_{ij} = \boldsymbol{\xi}_i \cdot \boldsymbol{\xi}_j = (\xi_i^a \boldsymbol{\sigma}_a) \cdot (\xi_j^b \boldsymbol{\sigma}_b)$, so that $\xi_i^a \xi_j^b \eta_{ab} = g_{ij}$. In general one has for both, the Clifford star product of the ambient space and the intrinsic Clifford star product (3.11):

$$\boldsymbol{\xi}_i *_C \boldsymbol{\xi}_j = g_{ij} + \boldsymbol{\xi}_i \boldsymbol{\xi}_j, \qquad \boldsymbol{\xi}_i *_C \boldsymbol{\xi}^j = \delta_i^j + \boldsymbol{\xi}_i \boldsymbol{\xi}^j, \qquad \text{and} \qquad \boldsymbol{\xi}^i *_C \boldsymbol{\xi}^j = g^{ij} + \boldsymbol{\xi}^i \boldsymbol{\xi}^j. \tag{3.12}$$

For an orientable manifold there exists a global unit-pseudoscalar $I_{(d)}(\boldsymbol{x}) = \boldsymbol{\xi}_1 \boldsymbol{\xi}_2 \dots \boldsymbol{\xi}_d / |\boldsymbol{\xi}_1 \boldsymbol{\xi}_2 \dots \boldsymbol{\xi}_d|$, which allows to define a projector P on the vector manifold that projects an arbitrary multivector $A(\boldsymbol{x})$ in the ambient space onto the vector manifold:

$$P(A(\boldsymbol{x}), \boldsymbol{x}) = (A(\boldsymbol{x}) \cdot I_{(d)}(\boldsymbol{x})) *_C I_{(d)}^{-1*_C}(\boldsymbol{x}). \tag{3.13}$$

A vector $\boldsymbol{v}(\boldsymbol{x}) = v^a(x^i)\boldsymbol{\sigma}_a$ in a point of the vector manifold can then be decomposed into an intrinsic part $P(\boldsymbol{v}) = (\boldsymbol{\xi}_i \cdot \boldsymbol{v})\boldsymbol{\xi}^i = (v_a \xi_i^a)\boldsymbol{\xi}^i$ which is tangent to the manifold and an extrinsic part $P_\perp(\boldsymbol{v}) = \boldsymbol{v} - P(\boldsymbol{v})$. Applying the projector to the nabla operator of the ambient space gives a vector derivative intrinsic to the manifold:

$$\boldsymbol{\partial} = P(\boldsymbol{\nabla}) = \boldsymbol{\xi}^i(\boldsymbol{\xi}_i \cdot \boldsymbol{\nabla}) = \boldsymbol{\xi}^i(\xi_i^a \partial_a) = \boldsymbol{\xi}^i \partial_i \tag{3.14}$$

and for a tangent vector $\boldsymbol{a}$ the directional derivative in the $\boldsymbol{a}$-direction is $\boldsymbol{a} \cdot \boldsymbol{\partial} = a^i \partial_i = a^i \xi_i^a \partial_a = \boldsymbol{a} \cdot \boldsymbol{\nabla}$. With the intrinsic vector derivative (3.14) the cotangent frame vectors $\boldsymbol{\xi}^i$ can also be obtained as the gradient of the coordinate functions $x^i(\boldsymbol{x})$ that arise from the inversion of the vector manifold parametrization $\boldsymbol{x} = \boldsymbol{x}(x^i)$:

$$\boldsymbol{\xi}^i = \boldsymbol{\partial} x^i. \tag{3.15}$$

If one now applies the directional derivative $\boldsymbol{a} \cdot \boldsymbol{\partial}$ on a tangent multivector field $A(\boldsymbol{x})$ the result does not in general lie completely inside the manifold. So if one wants to have a purely intrinsic result one has to use the projection operator P again. This leads to the definition of a new type of derivative that acts on tangent multivector fields and returns tangent multivector fields. This new derivative is the covariant derivative and is defined by:

$$(\boldsymbol{a} \cdot \boldsymbol{D})A(\boldsymbol{x}) = P\big((\boldsymbol{a} \cdot \boldsymbol{\partial})A(\boldsymbol{x})\big). \tag{3.16}$$

In the case of a scalar field $f(\boldsymbol{x})$ on the manifold the covariant and the intrinsic derivative are the same:

$$(\boldsymbol{a} \cdot \boldsymbol{\partial}) f = (\boldsymbol{a} \cdot \boldsymbol{D}) f, \tag{3.17}$$

while for tangent vector fields $\boldsymbol{a}$ and $\boldsymbol{b}$ one has

$$(\boldsymbol{a} \cdot \boldsymbol{\partial})\boldsymbol{b} = P\big((\boldsymbol{a} \cdot \boldsymbol{\partial})\boldsymbol{b}\big) + P_{\perp}\big((\boldsymbol{a} \cdot \boldsymbol{\partial})\boldsymbol{b}\big) = (\boldsymbol{a} \cdot \boldsymbol{D})\boldsymbol{b} + \boldsymbol{b} \cdot \mathtt{S}(\boldsymbol{a}), \tag{3.18}$$

where $\mathtt{S}(\boldsymbol{a})$ is the so called shape tensor, which is a bivector that describes both intrinsic and extrinsic properties of the vector manifold. The multivector generalization of (3.18) is

$$(\boldsymbol{a} \cdot \boldsymbol{\partial}) A = (\boldsymbol{a} \cdot \boldsymbol{D}) A + A \times \mathtt{S}(\boldsymbol{a}), \tag{3.19}$$

where $A \times B = \frac{1}{2}(A *_C B - B *_C A) = \frac{1}{2}[A, B]_{*_C}$ is the commutator product (not to be confused with the vector cross product used in (2.9); the cross product of two three-dimensional vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ and the commutator product of the corresponding bivectors $\mathtt{A} = I_{(3)} *_C \boldsymbol{a}$ and $\mathtt{B} = I_{(3)} *_C \boldsymbol{b}$ are connected according to $-I_{(3)} *_C (\boldsymbol{a} \times \boldsymbol{b}) = \frac{1}{2}\left[I_{(3)} *_C \boldsymbol{a}, I_{(3)} *_C \boldsymbol{b}\right]_{*_C} = \mathtt{A} \times \mathtt{B}$). The commutator product of an r-vector and a bivector gives again an r-vector so that all terms in (3.19) are r-vectors. Furthermore it is clear that (3.19) reduces to (3.18) if A is a vector field and to (3.17) if A is a scalar field.

As a tangent vector $(\boldsymbol{a} \cdot \boldsymbol{D})\boldsymbol{b}$ can be expanded in the $\boldsymbol{\xi}_i$ base:

$$(\boldsymbol{a} \cdot \boldsymbol{D})\boldsymbol{b} = a^j \big((D_j b^i)\boldsymbol{\xi}_i + b^i (D_j \boldsymbol{\xi}_i)^k \boldsymbol{\xi}_k\big) = a^j \big(\partial_j b^i + b^k \Gamma^i_{jk}\big)\boldsymbol{\xi}_i, \tag{3.20}$$

where $\Gamma^i_{jk} = (D_j \boldsymbol{\xi}_k) \cdot \boldsymbol{\xi}^i = \big(D_j \boldsymbol{\xi}_k\big)^i$ is the i-th component of $D_j \boldsymbol{\xi}_k$, which extrinsically can be written as $\Gamma^i_{jk} = \big(D_j \xi^a_k \boldsymbol{\sigma}_a\big) \cdot \xi^i_b \boldsymbol{\sigma}^b = (\partial_j \xi^a_k)\xi^i_a$. One of the properties the Γ^k_{ij} fulfill is the metric compatibility $\partial_k g_{ij} - \Gamma^l_{ki} g_{lj} - \Gamma^l_{kj} g_{li} = 0$, which can be found if one applies D_k on both sides of $\boldsymbol{\xi}_i \cdot \boldsymbol{\xi}_j = g_{ij}$. This means that the Γ^i_{jk} are the Christoffel symbols and $(\boldsymbol{a} \cdot \boldsymbol{D})\boldsymbol{b}$ is the Levi-Civita connection. The symmetry of the Γ^i_{jk} in the lower indices is a consequence of

$$\partial_i \boldsymbol{\xi}_j - \partial_j \boldsymbol{\xi}_i = (\partial_i \partial_j - \partial_j \partial_i)\boldsymbol{x} = 0. \tag{3.21}$$

Projecting into the manifold gives $D_i \boldsymbol{\xi}_j - D_j \boldsymbol{\xi}_i = 0$, so that the symmetry of the Γ^i_{jk} in the lower indices follows. From (3.21) follows further, that

$$(\boldsymbol{a} \cdot \boldsymbol{\partial})\boldsymbol{b} - (\boldsymbol{b} \cdot \boldsymbol{\partial})\boldsymbol{a} = \big(a^j(\partial_j b^i) - b^j(\partial_j a^i)\big)\boldsymbol{\xi}_i \tag{3.22}$$

is an intrinsic quantity that corresponds to the Lie derivative or the Jacobi-Lie bracket

$$\mathscr{L}_{\boldsymbol{a}} \boldsymbol{b} = [\boldsymbol{a}, \boldsymbol{b}]_{JLB} \equiv (\boldsymbol{a} \cdot \boldsymbol{\partial})\boldsymbol{b} - (\boldsymbol{b} \cdot \boldsymbol{\partial})\boldsymbol{a} = (\boldsymbol{a} \cdot \boldsymbol{D})\boldsymbol{b} - (\boldsymbol{b} \cdot \boldsymbol{D})\boldsymbol{a}. \tag{3.23}$$

The holonomy condition (3.21) can then be written with $\boldsymbol{\xi}_i \cdot \boldsymbol{\partial} = \partial_i$ in the more familiar form $\mathscr{L}_{\boldsymbol{\xi}_i} \boldsymbol{\xi}_j = \big[\boldsymbol{\xi}_i, \boldsymbol{\xi}_j\big]_{JLB} = 0$. One can also conclude with (3.18) that since $[\boldsymbol{a}, \boldsymbol{b}]_{JLB}$ is an intrinsic quantity, the extrinsic parts in the Jacobi-Lie bracket have to cancel, i.e. $\boldsymbol{a} \cdot \mathtt{S}(\boldsymbol{b}) = \boldsymbol{b} \cdot \mathtt{S}(\boldsymbol{a})$.

A natural generalization of the Lie derivative to multivectors is given by the Schouten-Nijenhuis bracket

$$\mathscr{L}_{A_{(r)}} B_{(s)} = \big[A_{(r)}, B_{(s)}\big]_{SNB} = (-1)^{r-1}(A_{(r)} \cdot \boldsymbol{D}) B_{(s)} + (-1)^{rs}(-1)^{s-1}(B_{(s)} \cdot \boldsymbol{D}) A_{(r)}. \tag{3.24}$$

The Schouten-Nijenhuis bracket can be written in this way due to the fact that (3.24) has the grade $r+s-1$, fulfills

$$\left[A_{(r)},B_{(s)}\right]_{SNB} = (-1)^{rs}\left[B_{(s)},A_{(r)}\right]_{SNB}, \tag{3.25}$$

$$\left[A_{(r)},B_{(s)}C_{(t)}\right]_{SNB} = \left[A_{(r)},B_{(s)}\right]_{SNB}C_{(t)} + (-1)^{rs+s}B_{(s)}\left[A_{(r)},C_{(t)}\right]_{SNB} \tag{3.26}$$

and reduces for scalar functions f, g and vector fields $\boldsymbol{a}$ and $\boldsymbol{b}$ to

$$[f,g]_{SNB}=0, \qquad [\boldsymbol{a},f]_{SNB}=(\boldsymbol{a}\cdot\boldsymbol{D})f \qquad \text{and} \qquad [\boldsymbol{a},\boldsymbol{b}]_{SNB}=\mathscr{L}_{\boldsymbol{a}}\boldsymbol{b}. \tag{3.27}$$

Furthermore one has the super-Jacobi-identity

$$(-1)^{rt}\left[\left[A_{(r)},B_{(s)}\right]_{SNB},C_{(t)}\right]_{SNB} + (-1)^{rs}\left[\left[B_{(s)},C_{(t)}\right]_{SNB},A_{(r)}\right]_{SNB} + (-1)^{st}\left[\left[C_{(t)},A_{(r)}\right]_{SNB},B_{(s)}\right]_{SNB} = 0. \tag{3.28}$$

4 Exterior Calculus

The exterior calculus [11] can be constructed by noting that the cotangent frame vector or one-form (3.15) can be written with (3.17) as

$$\boldsymbol{\xi}^i = \boldsymbol{D}x^i = \boldsymbol{\partial}x^i \equiv \boldsymbol{d}x^i. \tag{4.29}$$

In order to see how the directional covariant derivative acts on a general one-form $\boldsymbol{\omega}=\omega_i\boldsymbol{\xi}^i$ one first applies D_j on both sides of $\boldsymbol{\xi}^i\cdot\boldsymbol{\xi}_k=\delta^i_k$ which gives $(D_j\boldsymbol{\xi}^i)\cdot\boldsymbol{\xi}_k=(D_j\boldsymbol{\xi}^i)_k=-\Gamma^i_{jk}$, so that the covariant derivative of $\boldsymbol{\omega}$ reads

$$(\boldsymbol{a}\cdot\boldsymbol{D})\boldsymbol{\omega} = a^j\big((D_j\omega_i)\boldsymbol{\xi}^i+\omega_i(D_j\boldsymbol{\xi}^i)_k\boldsymbol{\xi}^k\big) = a^j\big(\partial_j\omega_i-\omega_k\Gamma^k_{ji}\big)\boldsymbol{\xi}^i. \tag{4.30}$$

Furthermore it is easy to see that $\boldsymbol{dd}x^i=\boldsymbol{D}\boldsymbol{\xi}^i=0$. The closedness of $\boldsymbol{\xi}^i$ can be used to calculate the relation of the Γ^i_{jk} and the metric:

$$\begin{aligned}
\Gamma^i_{jk}=(D_j\boldsymbol{\xi}_k)\cdot\boldsymbol{\xi}^i &= \frac{1}{2}\left[(D_j\boldsymbol{\xi}_k)+(D_k\boldsymbol{\xi}_j)\right]\cdot\boldsymbol{\xi}^i && (4.31)\\
&= \frac{1}{2}\left[\boldsymbol{\xi}_j\cdot(\boldsymbol{D}\boldsymbol{\xi}_k)+\Gamma^l_{mk}g_{jl}\boldsymbol{\xi}^m+\boldsymbol{\xi}_k\cdot(\boldsymbol{D}\boldsymbol{\xi}_j)+\Gamma^l_{mj}g_{kl}\boldsymbol{\xi}^m\right]\cdot\boldsymbol{\xi}^i && (4.32)\\
&= \frac{1}{2}\left[\boldsymbol{\xi}_j\cdot(\boldsymbol{D}g_{km}\boldsymbol{\xi}^m)+\boldsymbol{\xi}_k\cdot(\boldsymbol{D}g_{jm}\boldsymbol{\xi}^m)+(\partial_m g_{jk})\boldsymbol{\xi}^m\right]\cdot\boldsymbol{\xi}^i && (4.33)\\
&= \frac{1}{2}\left[(\partial_n g_{km})\boldsymbol{\xi}_j\cdot\boldsymbol{\xi}^n\boldsymbol{\xi}^m+(\partial_n g_{jm})\boldsymbol{\xi}_k\cdot\boldsymbol{\xi}^n\boldsymbol{\xi}^m+(\partial_m g_{jk})\boldsymbol{\xi}^m\right]\cdot\boldsymbol{\xi}^i && (4.34)\\
&= \frac{1}{2}\left[(\partial_n g_{km})(\delta^n_j\boldsymbol{\xi}^m-\delta^m_j\boldsymbol{\xi}^n)+(\partial_n g_{jm})(\delta^n_k\boldsymbol{\xi}^m-\delta^m_k\boldsymbol{\xi}^n)+(\partial_m g_{jk})\boldsymbol{\xi}^m\right]\cdot\boldsymbol{\xi}^i && (4.35)\\
&= \frac{1}{2}g^{il}\left[\partial_j g_{kl}+\partial_k g_{jl}-\partial_l g_{jk}\right], && (4.36)
\end{aligned}$$

where one uses in (4.31)

$$\boldsymbol{\xi}_j\cdot(\boldsymbol{D}\boldsymbol{\xi}_k)=\boldsymbol{\xi}_j\cdot(\boldsymbol{\xi}^iD_i\boldsymbol{\xi}_k)=\boldsymbol{\xi}_j\cdot(\boldsymbol{\xi}^i\Gamma^l_{ik}\boldsymbol{\xi}_l)=\Gamma^l_{ik}(\delta^i_j\boldsymbol{\xi}_l-g_{jl}\boldsymbol{\xi}^i)=D_j\boldsymbol{\xi}_k-\Gamma^l_{ik}g_{jl}\boldsymbol{\xi}^i. \tag{4.37}$$

With expression (4.36) it is possible to show that the shape bivector can be written as

$$\mathtt{S}(\boldsymbol{a}) = \frac{1}{2}\big(\boldsymbol{\xi}^i \boldsymbol{\partial} a_i - \boldsymbol{\xi}_i \boldsymbol{\partial} a^i + \boldsymbol{\xi}^i (\boldsymbol{a}\cdot\boldsymbol{\partial})\boldsymbol{\xi}_i\big), \tag{4.38}$$

or $\mathtt{S}_i = \mathtt{S}(\boldsymbol{\xi}_i) = \frac{1}{2}\boldsymbol{\xi}^j \boldsymbol{\xi}^k \partial_k g_{ij} + \frac{1}{2}\boldsymbol{\xi}^j \partial_i \boldsymbol{\xi}_j$. This can be proved by calculating

$$\begin{aligned}
\boldsymbol{b}\cdot\mathtt{S}(\boldsymbol{a}) &= \frac{1}{2}\big(b^i(\partial_j a_i)\boldsymbol{\xi}^j - b^j(\partial_j a_i)\boldsymbol{\xi}^i - b_i(\partial_j a^i)\boldsymbol{\xi}^j + b^j(\partial_j a^i)\boldsymbol{\xi}_i + a^j b^i(\partial_j \boldsymbol{\xi}_i) - a^j b^k \boldsymbol{\xi}^i(\boldsymbol{\xi}_k\cdot\partial_j\boldsymbol{\xi}_i)\big) && (4.39)\\
&= \frac{1}{2}\big(a^k b^i(\partial_j g_{ik})\boldsymbol{\xi}^j - a^k b^i(\partial_i g_{jk})\boldsymbol{\xi}^j + a^i b^j(\partial_i \boldsymbol{\xi}_j) - a^j b^k \boldsymbol{\xi}^i(\boldsymbol{\xi}_k\cdot\partial_j\boldsymbol{\xi}_i)\big) && (4.40)\\
&= -a^k b^i \Gamma^l_{ki}\boldsymbol{\xi}_l + \frac{1}{2}a^j b^k \boldsymbol{\xi}^i(\boldsymbol{\partial} g_{ki}) + \frac{1}{2}a^i b^j(\partial_i\boldsymbol{\xi}_j) - \frac{1}{2}a^j b^k \boldsymbol{\xi}^i(\boldsymbol{\xi}_k\cdot\partial_j\boldsymbol{\xi}_i) && (4.41)\\
&= -a^k b^i \Gamma^l_{ki}\boldsymbol{\xi}_l + \frac{1}{2}a^i b^j \boldsymbol{\xi}^k\left[(\partial_i\boldsymbol{\xi}_j)\cdot\boldsymbol{\xi}_k\right] + \frac{1}{2}a^i b^j(\partial_i\boldsymbol{\xi}_j) && (4.42)\\
&= a^i b^j(\partial_i\boldsymbol{\xi}_j) - a^i b^j \Gamma^k_{ij}\boldsymbol{\xi}_k && (4.43)\\
&= (\boldsymbol{a}\cdot\boldsymbol{\partial})\boldsymbol{b} - (\boldsymbol{a}\cdot\boldsymbol{D})\boldsymbol{b} && (4.44)\\
&= P_\perp\big((\boldsymbol{a}\cdot\boldsymbol{\partial})\boldsymbol{b}\big), && (4.45)
\end{aligned}$$

which corresponds to definition (3.18). In (4.40) relation (4.36) was used and in (4.42) one uses

$$\boldsymbol{\xi}^k\left[(\partial_i\boldsymbol{\xi}_j)\cdot\boldsymbol{\xi}_k\right] = \xi^k_c\boldsymbol{\sigma}^c\left[(\partial_i\xi^a_j\boldsymbol{\sigma}_a)\cdot\xi^b_k\boldsymbol{\sigma}_b\right] = (\partial_i\xi^a_j)\boldsymbol{\sigma}_a = \partial_i\boldsymbol{\xi}_j. \tag{4.46}$$

While the exterior derivative of the reciprocal basis vectors is zero, the exterior derivative of a general one-form $\boldsymbol{\omega} = \omega_i\boldsymbol{\xi}^i$ is a two-form $\boldsymbol{d\omega} = (\boldsymbol{D}\omega_j)\boldsymbol{\xi}^j + \omega_j\boldsymbol{D}\boldsymbol{\xi}^j = (\partial_i\omega_j)\boldsymbol{\xi}^i\boldsymbol{\xi}^j$. A general r-form is then a covariant r-blade $A^{(r)}$ [11] and can be written as

$$A^{(r)} = \frac{1}{r!}A_{i_1 i_2\ldots i_r}\boldsymbol{d}x^{i_1}\boldsymbol{d}x^{i_2}\ldots\boldsymbol{d}x^{i_r} = \frac{1}{r!}A_{i_1 i_2\ldots i_r}\boldsymbol{\xi}^{i_1}\boldsymbol{\xi}^{i_2}\ldots\boldsymbol{\xi}^{i_r}. \tag{4.47}$$

Applying the exterior differential, to $A^{(r)}$ gives

$$\boldsymbol{d}A^{(r)} = \frac{1}{r!}\left(\frac{\partial A_{i_1 i_2\ldots i_r}}{\partial x^j}\right)\boldsymbol{d}x^j\boldsymbol{d}x^{i_1}\boldsymbol{d}x^{i_2}\ldots\boldsymbol{d}x^{i_r} = \frac{1}{r!}\left(\frac{\partial A_{i_1 i_2\ldots i_r}}{\partial x^j}\right)\boldsymbol{\xi}^j\boldsymbol{\xi}^{i_1}\boldsymbol{\xi}^{i_2}\ldots\boldsymbol{\xi}^{i_r}, \tag{4.48}$$

which is a $(r+1)$-form or a covariant $(r+1)$-blade.

It is then also straight forward to translate other structures of exterior calculus into the language of superanalytic geometric algebra, for example the Hodge dual is given by

$$\star\left(\boldsymbol{\xi}^{i_1}\boldsymbol{\xi}^{i_2}\ldots\boldsymbol{\xi}^{i_r}\right) = \frac{\sqrt{|g|}}{(d-r)!}\varepsilon^{i_1\ldots i_r}{}_{i_{r+1}\ldots i_d}\boldsymbol{\xi}^{i_{r+1}}\ldots\boldsymbol{\xi}^{i_d}, \tag{4.49}$$

with $\varepsilon^{i_1\ldots i_r}{}_{i_{r+1}\ldots i_d} = g^{i_1 j_1}\ldots g^{i_r j_r}\varepsilon_{j_1\ldots j_r i_{r+1}\ldots i_d}$ and $\varepsilon_{i_1\ldots i_d} = 1$ for even permutations. In the euclidian or Minkowski case the Hodge dual can be written as

$$\star A^{(r)} = (-1)^{(d-r)r+r(r-1)/2}I^{(d)} *_C A^{(r)} \tag{4.50}$$

and the inverse Hodge star operator in the euclidian case as

$$\star^{-1} A^{(r)} = (-1)^{r(d-r)} \star A^{(r)} = (-1)^{(r-1)r/2} I^{(d)} *_C A^{(r)}, \tag{4.51}$$

while in the four dimensional Minkowski case one has an additional minus sign, i.e. $\star^{-1} = (-1)^{r(d-r)+1}\star$. With the Hodge star operator as defined in (4.49) the coderivative $\boldsymbol{d}^\dagger$ is given in the Riemannian case as

$$\boldsymbol{d}^\dagger A^{(r)} = (-1)^{dr+d+1} \star \boldsymbol{d} \star A^{(r)} \tag{4.52}$$

and in the Minkowski case as $\boldsymbol{d}^\dagger A^{(r)} = (-1)^{dr+d} \star \boldsymbol{d} \star A^{(r)}$. Writing this down in components one sees that the coderivative maps an r-form into an $(r-1)$-form and can be written as $\boldsymbol{d}^\dagger A^{(r)} = -\boldsymbol{d} \cdot A^{(r)}$.

The interior product that maps an r-blade $A^{(r)}$ into an $(r-1)$-blade is just the scalar product with a vector, which can be generalized to the case of two multivectors $A_{(r)}$ and $B^{(s)}$ as

$$i_{A_{(r)}} B^{(s)} = \overline{A_{(r)}} \cdot B^{(s)}, \tag{4.53}$$

so that one has for example

$$\begin{aligned}(\overline{\boldsymbol{a}_1\boldsymbol{a}_2\ldots\boldsymbol{a}_{r+1}})\cdot \boldsymbol{d}A^{(r)} &= \sum_{n=1}^{r+1}(-1)^{n+1}(\boldsymbol{a}_n\cdot\boldsymbol{\partial})\left(\overline{\boldsymbol{a}_1\ldots\check{\boldsymbol{a}}_n\ldots\boldsymbol{a}_{r+1}}\right)\cdot A^{(r)} \\ &\quad + \sum_{m<n}(-1)^{m+n}\left(\overline{[\boldsymbol{a}_m,\boldsymbol{a}_n]_{JLB}\,\boldsymbol{a}_1\ldots\check{\boldsymbol{a}}_m\ldots\check{\boldsymbol{a}}_n\ldots\boldsymbol{a}_{r+1}}\right)\cdot A^{(r)}.\end{aligned} \tag{4.54}$$

In the same way Cartan's magic formula

$$\mathscr{L}_{\boldsymbol{a}}\boldsymbol{\omega} = (\boldsymbol{d}i_{\boldsymbol{a}} + i_{\boldsymbol{a}}\boldsymbol{d})\,\boldsymbol{\omega} = \left(a^i(\partial_i\omega_j) + (\partial_j a^i)\omega_i\right)\boldsymbol{\xi}^j, \tag{4.55}$$

generalizes to

$$\mathscr{L}_{\boldsymbol{a}}A^{(r)} = (\boldsymbol{d}i_{\boldsymbol{a}} + i_{\boldsymbol{a}}\boldsymbol{d})\,A^{(r)} = \boldsymbol{D}(\boldsymbol{a}\cdot A^{(r)}) + \boldsymbol{a}\cdot(\boldsymbol{D}A^{(r)}). \tag{4.56}$$

Up to now only the coordinate basis of the $\boldsymbol{\xi}_i$ was used, in general it is also possible to use a non-coordinate basis given by

$$\boldsymbol{\vartheta}_r = \vartheta^i_r\boldsymbol{\xi}_i \qquad \text{and} \qquad \boldsymbol{\xi}_i = \vartheta^r_i\boldsymbol{\vartheta}_r, \tag{4.57}$$

where $\vartheta^i_r = \boldsymbol{\vartheta}_r\cdot\boldsymbol{\xi}^i$ are functions of the x^k, with $\vartheta^r_i\vartheta^j_r = \delta^j_i$ and $g_{ij} = \vartheta^r_i\vartheta^s_j g_{rs}$. Analogously the reciprocal non-coordinate basis $\boldsymbol{\vartheta}^r$ can be expanded with the ϑ^r_i in the reciprocal coordinate basis of the $\boldsymbol{\xi}^i$. A special choice for the non-coordinate frame fields is obtained by the conditions $\boldsymbol{\vartheta}_r\cdot\boldsymbol{\vartheta}_s = \eta_{rs}$ and $\partial_i\boldsymbol{\vartheta}_r = 0$. This means the $\boldsymbol{\vartheta}_r$ span a (pseudo)-euclidian base and they move on the vector manifold so that

$$D_i\boldsymbol{\vartheta}_r = -\boldsymbol{\vartheta}_r\cdot\mathtt{S}_i. \tag{4.58}$$

This shows that the shape tensor, which has in the $\boldsymbol{\vartheta}_r$-frame the form $\mathtt{S}_r = \mathtt{S}(\boldsymbol{\vartheta}_r) = \vartheta^i_r\mathtt{S}_i$, is proportional to the Fock-Ivanenko bivector Γ_i [19], i.e. $\mathtt{S}_i = -2\Gamma_i$.

For general non-coordinate basis vectors the Jacobi-Lie bracket is no longer zero, one rather has

$$\begin{aligned}[\boldsymbol{\vartheta}_r,\boldsymbol{\vartheta}_s]_{JLB} &= \vartheta^i_r(\boldsymbol{\xi}_i\cdot\boldsymbol{D})(\vartheta^j_s\boldsymbol{\xi}_j) - \vartheta^i_s(\boldsymbol{\xi}_i\cdot\boldsymbol{D})(\vartheta^j_r\boldsymbol{\xi}_j) && (4.59)\\ &= \vartheta^i_r\left[(D_i\vartheta^j_s)\boldsymbol{\xi}_j + \vartheta^j_s(D_i\boldsymbol{\xi}_j)\right] - \vartheta^i_s\left[(D_i\vartheta^j_r)\boldsymbol{\xi}_j + \vartheta^j_r(D_i\boldsymbol{\xi}_j)\right] && (4.60)\\ &= \left[\vartheta^i_r D_i\vartheta^j_s - \vartheta^i_s D_i\vartheta^j_r\right]\boldsymbol{\xi}_j && (4.61)\\ &= \left[\partial_r\vartheta^j_s - \partial_s\vartheta^j_r\right]\vartheta^t_j\boldsymbol{\vartheta}_t && (4.62)\\ &= C^t_{rs}\boldsymbol{\vartheta}_t, && (4.63)\end{aligned}$$

with $C^t_{rs} = [\boldsymbol{\vartheta}_r, \boldsymbol{\vartheta}_s]_{JLB} \cdot \boldsymbol{\vartheta}^t = \left[\partial_r \vartheta^j_s - \partial_s \vartheta^j_r\right] \vartheta^t_j$. For tangent vector fields $\boldsymbol{a} = a^r \boldsymbol{\vartheta}_r$ and $\boldsymbol{b} = b^s \boldsymbol{\vartheta}_s$, it follows then that

$$\mathscr{L}_{\boldsymbol{a}} \boldsymbol{b} = [\boldsymbol{a}, \boldsymbol{b}]_{JLB} = \left(a^r(\partial_r b^s) - b^r(\partial_r a^s)\right)\boldsymbol{\vartheta}_s + a^r b^s \left[\boldsymbol{\vartheta}_r, \boldsymbol{\vartheta}_s\right]_{JLB}, \tag{4.64}$$

which reduces in a coordinate basis to (3.22).

In the non-coordinate basis a straight forward calculation shows that the Γ^t_{rs} are given by

$$\Gamma^t_{rs} = -\left[(\boldsymbol{\vartheta}_r \cdot \boldsymbol{D})\boldsymbol{\vartheta}^t\right] \cdot \boldsymbol{\vartheta}_s = \frac{1}{2} g^{tu} \left[\partial_r g_{su} + \partial_s g_{ru} - \partial_u g_{rs}\right] + \frac{1}{2} g^{tu}(C_{urs} + C_{usr} - C_{sru}), \tag{4.65}$$

where $C_{rsu} = g_{tu} C^t_{rs}$. While in the coordinate base $\left[\boldsymbol{\xi}_i, \boldsymbol{\xi}_j\right]_{JLB} = 0$ insured that the Γ^k_{ij} are symmetric in the lower indices, one has in the non-coordinate basis the relation $\Gamma^t_{rs} - \Gamma^t_{sr} = C^t_{rs}$. This implies that the non-coordinate one-forms $\boldsymbol{\vartheta}^r$ are not closed:

$$\begin{aligned}
\boldsymbol{d}\boldsymbol{\vartheta}^r = \boldsymbol{\xi}^j D_j(\vartheta^r_i \boldsymbol{\xi}^i) &= \frac{1}{2}(\partial_i \vartheta^r_j - \partial_j \vartheta^r_i)\boldsymbol{\xi}^i \boldsymbol{\xi}^j && (4.66)\\
&= \frac{1}{2}\left(\vartheta^s_i(\boldsymbol{\vartheta}_s \cdot \boldsymbol{\partial})\vartheta^r_j - \vartheta^s_j(\boldsymbol{\vartheta}_s \cdot \boldsymbol{\partial})\vartheta^r_i\right) \vartheta^i_t \vartheta^j_u \boldsymbol{\vartheta}^t \boldsymbol{\vartheta}^u && (4.67)\\
&= \frac{1}{2}\left(\vartheta^i_u(\boldsymbol{\vartheta}_t \cdot \boldsymbol{\partial})\vartheta^r_i - \vartheta^j_t(\boldsymbol{\vartheta}_u \cdot \boldsymbol{\partial})\vartheta^r_j\right) \boldsymbol{\vartheta}^t \boldsymbol{\vartheta}^u && (4.68)\\
&= -\frac{1}{2}\left(\vartheta^r_i(\boldsymbol{\vartheta}_t \cdot \boldsymbol{\partial})\vartheta^i_u - \vartheta^r_j(\boldsymbol{\vartheta}_u \cdot \boldsymbol{\partial})\vartheta^j_t\right) \boldsymbol{\vartheta}^t \boldsymbol{\vartheta}^u && (4.69)\\
&= -\frac{1}{2} C^r_{tu} \boldsymbol{\vartheta}^t \boldsymbol{\vartheta}^u, && (4.70)
\end{aligned}$$

which is the Maurer-Cartan equation. The exterior derivative of a general non-coordinate one-form $\boldsymbol{\alpha} = \alpha_r \boldsymbol{\vartheta}^r$ is

$$\boldsymbol{d}\boldsymbol{\alpha} = (\boldsymbol{D}\alpha_r)\boldsymbol{\vartheta}^r + \alpha_r \boldsymbol{D}\boldsymbol{\vartheta}^r = (\partial_r \alpha_s - \alpha_t \Gamma^t_{rs})\boldsymbol{\vartheta}^r \boldsymbol{\vartheta}^s, \tag{4.71}$$

for the exterior derivative of a general r-form in the non-coordinate basis $A^{(r)} = \frac{1}{r!} A_{s_1 \ldots s_r} \boldsymbol{\vartheta}^{s_1} \ldots \boldsymbol{\vartheta}^{s_r}$ one obtains

$$\boldsymbol{d}A^{(r)} = \frac{(-1)^r}{(r+1)!}\left(\partial_{[s_{r+1}} A_{s_1 \ldots s_r]} - \Gamma^t_{[s_{r+1} s_k} A_{s_1 \ldots s_{k-1} t s_{k+1} \ldots s_r]}\right) \boldsymbol{\vartheta}^{s_1} \boldsymbol{\vartheta}^{s_2} \ldots \boldsymbol{\vartheta}^{s_{r+1}}, \tag{4.72}$$

where the square brackets antisymmetrize the lower indices.

The formalism developed so far can also be used to describe tensor calculus. A tensor is a multilinear map of r vectors and s one-forms into the real numbers and can be written as

$$\mathsf{T} = T^{i_1 \ldots i_r}_{j_1 \ldots j_s} \boldsymbol{\xi}_{i_1} \otimes \cdots \otimes \boldsymbol{\xi}_{i_r} \otimes \boldsymbol{\xi}^{j_1} \otimes \cdots \otimes \boldsymbol{\xi}^{j_s}. \tag{4.73}$$

The components of the tensor are obtained as

$$T^{i_1 \ldots i_r}_{j_1 \ldots j_s} = \mathsf{T}(\boldsymbol{\xi}^{i_1}, \ldots, \boldsymbol{\xi}^{i_r}, \boldsymbol{\xi}_{j_1}, \ldots, \boldsymbol{\xi}_{j_s}) = i_{\boldsymbol{\xi}^{i_1} \otimes \cdots \otimes \boldsymbol{\xi}_{j_s}} \mathsf{T} = (\boldsymbol{\xi}^{i_1} \otimes \cdots \otimes \boldsymbol{\xi}_{j_s}) \cdot \mathsf{T}. \tag{4.74}$$

For example the metric tensor $\mathsf{g} = g_{ij} \boldsymbol{\xi}^i \otimes \boldsymbol{\xi}^j = g_{ij} \boldsymbol{d}x^i \otimes \boldsymbol{d}x^j$ maps two vectors $\boldsymbol{a} = a^i \boldsymbol{\xi}_i$ and $\boldsymbol{b} = b^i \boldsymbol{\xi}_i$ into a scalar according to

$$\mathsf{g}(\boldsymbol{a}, \boldsymbol{b}) = i_{\boldsymbol{a} \otimes \boldsymbol{b}} \mathsf{g} = (a^k \boldsymbol{\xi}_k \otimes b^l \boldsymbol{\xi}_l) \cdot (g_{ij} \boldsymbol{\xi}^i \otimes \boldsymbol{\xi}^j) = g_{ij} a^i b^j. \tag{4.75}$$

The above tensor concept can be generalized in several ways. For example one can consider a function that maps r contravariant and s covariant blades of arbitrary grade into a scalar, i.e. tensors of the form

$$\mathsf{T} = T^{i_1 \dots i_r}_{j_1 \dots j_s} A^{(r_1)}_{i_1} \otimes \dots \otimes A^{(r_r)}_{i_r} \otimes B^{j_1}_{(s_1)} \otimes \dots \otimes B^{j_s}_{(s_s)}. \tag{4.76}$$

The other possibility is to consider multivector valued tensors. In this case a tensor maps a number of (multi)vectors into a multivector, that does not have to lie in the same vector space. All these possible generalizations will appear in the following.

5 Curvature and Torsion

Curvature can be described if one transports a vector around a closed path and measures the difference of the initial and the transported vector. The path can be thought of as spanned by two tangent vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ and closes by $[\boldsymbol{a}, \boldsymbol{b}]_{JLB}$. One can then act with a curvature operator on a tangent vector $\boldsymbol{c} = c^r \boldsymbol{\vartheta}_r$:

$$\begin{aligned}&[(\boldsymbol{a}\cdot\boldsymbol{D})(\boldsymbol{b}\cdot\boldsymbol{D}) - (\boldsymbol{b}\cdot\boldsymbol{D})(\boldsymbol{a}\cdot\boldsymbol{D}) - [\boldsymbol{a},\boldsymbol{b}]_{JLB}\cdot\boldsymbol{D}]\,\boldsymbol{c} \\ &\qquad = a^r b^s c^t \left(D_r D_s - D_s D_r - C^u_{rs} D_u\right)\boldsymbol{\vartheta}_t = a^r b^s c^t R^u_{rst}\boldsymbol{\vartheta}_u,\end{aligned} \tag{5.1}$$

with

$$\begin{aligned} R^u_{rst} &= \left[\big(D_r D_s - D_s D_r - [\boldsymbol{\vartheta}_r, \boldsymbol{\vartheta}_s]_{JLB}\cdot\boldsymbol{D}\big)\boldsymbol{\vartheta}_t\right]\cdot\boldsymbol{\vartheta}^u &(5.2)\\ &= [D_r(\Gamma^w_{st}\boldsymbol{\vartheta}_w) - D_s(\Gamma^w_{rt}\boldsymbol{\vartheta}_w) - C^w_{rs}(D_w\boldsymbol{\vartheta}_t)]\cdot\boldsymbol{\vartheta}^u &(5.3)\\ &= \partial_r\Gamma^u_{st} - \partial_s\Gamma^u_{rt} + \Gamma^u_{rw}\Gamma^w_{st} - \Gamma^u_{sw}\Gamma^w_{rt} + \Gamma^w_{rs}\Gamma^u_{wt} - \Gamma^w_{sr}\Gamma^u_{wt}, &(5.4)\end{aligned}$$

which in the case of a coordinate basis reduces to

$$R^l_{ijk} = \left[\big(D_i D_j - D_j D_i\big)\boldsymbol{\xi}_k\right]\cdot\boldsymbol{\xi}^l = \partial_i\Gamma^l_{jk} - \partial_j\Gamma^l_{ik} + \Gamma^l_{im}\Gamma^m_{jk} - \Gamma^l_{jm}\Gamma^m_{ik}. \tag{5.5}$$

Since the curvature operator maps three vectors into a fourth one, it can also be written as a tensor $\mathsf{R} = R^u_{rst}\boldsymbol{\vartheta}_u \otimes \boldsymbol{\vartheta}^r \otimes \boldsymbol{\vartheta}^s \otimes \boldsymbol{\vartheta}^t$. In general the curvature operator can act on a multivector A, so that one has with (3.19)

$$\begin{aligned}&\big[(\boldsymbol{a}\cdot\boldsymbol{D})(\boldsymbol{b}\cdot\boldsymbol{D}) - (\boldsymbol{b}\cdot\boldsymbol{D})(\boldsymbol{a}\cdot\boldsymbol{D}) - \big[\boldsymbol{a},\boldsymbol{b}\big]_{JLB}\cdot\boldsymbol{D}\big]A \\ &\qquad = \big[(\boldsymbol{a}\cdot\boldsymbol{\partial})\mathtt{S}(\boldsymbol{b}) - (\boldsymbol{b}\cdot\boldsymbol{\partial})\mathtt{S}(\boldsymbol{a}) + \mathtt{S}(\boldsymbol{a})\times\mathtt{S}(\boldsymbol{b}) - \mathtt{S}([\boldsymbol{a},\boldsymbol{b}]_{JLB})\big]\times A = \mathtt{R}(\boldsymbol{ab})\times A,\end{aligned} \tag{5.6}$$

which reduces to

$$[(\boldsymbol{a}\cdot\boldsymbol{D})(\boldsymbol{b}\cdot\boldsymbol{D}) - (\boldsymbol{b}\cdot\boldsymbol{D})(\boldsymbol{a}\cdot\boldsymbol{D}) - [\boldsymbol{a},\boldsymbol{b}]_{JLB}\cdot\boldsymbol{D}]\,\boldsymbol{c} = \mathtt{R}(\boldsymbol{ab})\cdot\boldsymbol{c} \tag{5.7}$$

acting on a vector. The bivector-valued function of a bivector

$$\mathtt{R}(\boldsymbol{ab}) = (\boldsymbol{a}\cdot\boldsymbol{\partial})\mathtt{S}(\boldsymbol{b}) - (\boldsymbol{b}\cdot\boldsymbol{\partial})\mathtt{S}(\boldsymbol{a}) + \mathtt{S}(\boldsymbol{a})\times\mathtt{S}(\boldsymbol{b}) - \mathtt{S}([\boldsymbol{a},\boldsymbol{b}]_{JLB}) \tag{5.8}$$

fulfills the Ricci and Bianchi identities

$$\begin{aligned} &\boldsymbol{a}\cdot\mathtt{R}(\boldsymbol{bc}) + \boldsymbol{b}\cdot\mathtt{R}(\boldsymbol{ca}) + \boldsymbol{c}\cdot\mathtt{R}(\boldsymbol{ab}) = 0 &(5.9)\\ \text{and}\qquad &(\boldsymbol{a}\cdot\boldsymbol{D})\mathtt{R}(\boldsymbol{bc}) + (\boldsymbol{b}\cdot\boldsymbol{D})\mathtt{R}(\boldsymbol{ca}) + (\boldsymbol{c}\cdot\boldsymbol{D})\mathtt{R}(\boldsymbol{ab}) = 0. &(5.10)\end{aligned}$$

Comparing (5.1) with (5.7) shows that the curvature is described by a bivector-valued function of a bivector according to

$$a^r b^s c^t R^u_{rst} \boldsymbol{\vartheta}_u = \mathtt{R}(\boldsymbol{ab}) \cdot \boldsymbol{c}. \tag{5.11}$$

But it is also possible to describe it by a scalar-valued function of a bivector, i.e. a two-form $R^u_t(\boldsymbol{ab}) = i_{\boldsymbol{ab}} R^u_t$ according to

$$a^r b^s c^t R^u_{rst} \boldsymbol{\vartheta}_u = c^t R^u_t(\boldsymbol{ab}) \boldsymbol{\vartheta}_u. \tag{5.12}$$

It is now easy to see from this definition and (5.4) that the curvature two-form R^u_t has the form

$$R^u_t = \left(\partial_v \Gamma^u_{wt} + \Gamma^u_{rt} \Gamma^r_{wv} + \Gamma^u_{vr} \Gamma^r_{wt}\right) \boldsymbol{\vartheta}^v \boldsymbol{\vartheta}^w, \tag{5.13}$$

which also can be expressed in another way. For this purpose one notices that the exterior derivative of $\boldsymbol{\vartheta}_r$ is a vector-valued one-form:

$$\boldsymbol{d}\boldsymbol{\vartheta}_r = \boldsymbol{\vartheta}^s D_s \boldsymbol{\vartheta}_r = \Gamma^t_{sr} \boldsymbol{\vartheta}^s \boldsymbol{\vartheta}_t = \boldsymbol{\omega}^t_r \boldsymbol{\vartheta}_t, \tag{5.14}$$

where $\boldsymbol{\omega}^t_r = \Gamma^t_{sr} \boldsymbol{\vartheta}^s$. With $\boldsymbol{\omega}^t_r$ the curvature two-form (5.13) can also be written as

$$R^u_t = \boldsymbol{d}\boldsymbol{\omega}^u_t + \boldsymbol{\omega}^u_r \boldsymbol{\omega}^r_t, \tag{5.15}$$

which is the first Cartan structure equation. Exterior differentiation of (5.15) gives the Bianchi identity for the curvature two-form: $\boldsymbol{d}R^r_s + \boldsymbol{\omega}^r_t R^t_s - R^r_t \boldsymbol{\omega}^t_s = 0$.

It is possible that the path spanned by two tangent vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ is not closed by $[\boldsymbol{a}, \boldsymbol{b}]_{JLB}$. This is measured by the torsion

$$(\boldsymbol{a} \cdot \boldsymbol{D})\boldsymbol{b} - (\boldsymbol{b} \cdot \boldsymbol{D})\boldsymbol{a} - [\boldsymbol{a}, \boldsymbol{b}]_{JLB} = a^r b^s T^t_{rs} \boldsymbol{\vartheta}_t \tag{5.16}$$

with

$$T^t_{rs} = \left[D_r \boldsymbol{\vartheta}_s - D_s \boldsymbol{\vartheta}_r - [\boldsymbol{\vartheta}_r, \boldsymbol{\vartheta}_s]_{JLB}\right] \cdot \boldsymbol{\vartheta}^t = \Gamma^t_{rs} - \Gamma^t_{sr} - C^t_{rs}, \tag{5.17}$$

which reduces in a coordinate basis to $T^k_{ij} = \Gamma^k_{ij} - \Gamma^k_{ji}$. This means that for non-vanishing torsion the Γ^k_{ij} are no longer symmetric in the lower indices so that $\boldsymbol{dd}x^i$ is no longer zero and the exterior differential of an r-form is given by

$$\begin{aligned} \boldsymbol{d}A^{(r)} = \boldsymbol{D}A^{(r)} &= \frac{1}{r!}\left(\frac{\partial A_{i_1 i_2 \ldots i_r}}{\partial x^j}\right) \boldsymbol{D}x^j \boldsymbol{D}x^{i_1} \boldsymbol{D}x^{i_2} \ldots \boldsymbol{D}x^{i_r} \\ &\quad + \frac{1}{r!} A_{i_1 i_2 \ldots i_r} \left[\boldsymbol{DD}x^{i_1} \boldsymbol{D}x^{i_2} \ldots \boldsymbol{D}x^{i_r} - \boldsymbol{D}x^{i_1} \boldsymbol{DD}x^{i_2} \boldsymbol{D}x^{i_3} \ldots \boldsymbol{D}x^{i_r} \right. \\ &\qquad \left. + \ldots + (-1)^{r-1} \boldsymbol{D}x^{i_1} \boldsymbol{D}x^{i_2} \ldots \boldsymbol{DD}x^{i_r}\right]. \end{aligned} \tag{5.18}$$

The torsion maps two vectors into a third one, so that it can also be written as a tensor $\mathsf{T} = T^t_{rs} \boldsymbol{\vartheta}_t \otimes \boldsymbol{\vartheta}^r \otimes \boldsymbol{\vartheta}^s$. The other possibility is to describe the torsion with a scalar-valued function of a bivector, i.e. a two-form $T^t(\boldsymbol{ab}) = i_{\boldsymbol{ab}} T^t$ according to

$$a^r b^s T^t_{rs} \boldsymbol{\vartheta}_t = T^t(\boldsymbol{ab}) \boldsymbol{\vartheta}_t. \tag{5.19}$$

It is then easy to see with (5.17) that the torsion two-form can be written as

$$T^t = \left(\Gamma^t_{rs} - \frac{1}{2} C^t_{sr}\right) \boldsymbol{\vartheta}^r \boldsymbol{\vartheta}^s. \tag{5.20}$$

With the Cartan one-form $\boldsymbol{\omega}^t_r$ this can also be written as

$$T^t = \boldsymbol{d}\boldsymbol{\vartheta}^t + \boldsymbol{\omega}^t_r \boldsymbol{\vartheta}^r, \tag{5.21}$$

which is the second Cartan structure equation. Applying the exterior differentiation on both sides of (5.21) gives the second Bianchi-identity $\boldsymbol{d}T^t + \boldsymbol{\omega}^t_r T^r = R^t_r \boldsymbol{\vartheta}^r$.

6 Rotor Groups and Bivector Algebras

The multivectors of even Grassmann grade are closed under the Clifford star product and form the group $Spin(p,q)$. An element $S \in Spin(p,q)$ fulfills $S *_C \overline{S} = \pm 1$ and a transformation $S *_C \boldsymbol{x} *_C S^{-1*_C}$ gives again a vector-valued result [5]. The elements $R \in Spin(p,q)$ with $R *_C \overline{R} = +1$ are called rotors and form the rotor group $Spin^+(p,q)$, which in the euclidian case is equal to the spin-group. For a rotor one has $R^{-1*_C} = \overline{R}$, so that a multivector A transforms as $R *_C A *_C \overline{R}$. A rotor can be written as a starexponential of a bivector, i.e. in general the rotor has for a bivector $\mathtt{B}$ the form

$$R(t) = \pm e_{*_C}^{\frac{t}{2}\mathtt{B}} \tag{6.1}$$

and the rotation of a vector $\boldsymbol{x}_0$ is given by $\boldsymbol{x}(t) = R(t) *_C \boldsymbol{x}_0 *_C \overline{R}(t)$. The bivector basis $\mathtt{B}_i$ of a rotor constitutes an algebra under the commutator product

$$\mathtt{B}_i \times \mathtt{B}_j = C_{ij}^k \mathtt{B}_k, \tag{6.2}$$

where the C_{ij}^k are the structure constants (note that one has here an additional factor $\frac{1}{2}$ due to the definition of the commutator product). Furthermore one can directly calculate

$$\kappa_{ij} = \mathtt{B}_i \cdot \mathtt{B}_j, \tag{6.3}$$

which is (proportional to) the Killing metric. As an example one can consider the group $SO(3)$. Given a three dimensional euclidian space with basis vectors $\boldsymbol{\sigma}_i$ the rotor is given by

$$R = R_0 + R_1 \boldsymbol{\sigma}_2 \boldsymbol{\sigma}_3 + R_2 \boldsymbol{\sigma}_3 \boldsymbol{\sigma}_1 + R_3 \boldsymbol{\sigma}_1 \boldsymbol{\sigma}_2, \tag{6.4}$$

with $R *_C \overline{R} = R_0^2 + R_1^2 + R_2^2 + R_3^2 = 1$, so that the rotor can also be parametrized with three parameters α, θ and φ as:

$$R(\alpha,\theta,\varphi) = \cos\alpha\cos\theta + \sin\alpha\cos\varphi \boldsymbol{\sigma}_2\boldsymbol{\sigma}_3 + \sin\alpha\sin\varphi \boldsymbol{\sigma}_3\boldsymbol{\sigma}_1 + \cos\alpha\sin\theta \boldsymbol{\sigma}_1\boldsymbol{\sigma}_2. \tag{6.5}$$

The three basis bivectors $\mathtt{B}_1 = \boldsymbol{\sigma}_2\boldsymbol{\sigma}_3$, $\mathtt{B}_2 = \boldsymbol{\sigma}_3\boldsymbol{\sigma}_1$ and $\mathtt{B}_3 = \boldsymbol{\sigma}_1\boldsymbol{\sigma}_2$ fulfill

$$\mathtt{B}_i \times \mathtt{B}_j = -\varepsilon_{ijk}\mathtt{B}_k \qquad \text{and} \qquad \kappa_{ij} = \mathtt{B}_i \cdot \mathtt{B}_j = -\delta_{ij}. \tag{6.6}$$

It is easy to see that the group vector manifold, which for $SO(3)$ is an S^3 embedded in a four dimensional euclidian space with basis vectors $\boldsymbol{\tau}_a$, can be read off from (6.5) as

$$\boldsymbol{r}_R(\alpha,\theta,\varphi) = \cos\alpha\cos\theta\,\boldsymbol{\tau}_1 + \sin\alpha\cos\varphi\,\boldsymbol{\tau}_2 + \sin\alpha\sin\varphi\,\boldsymbol{\tau}_3 + \cos\alpha\sin\theta\,\boldsymbol{\tau}_4. \tag{6.7}$$

The rotors act on themselves by left- and right-translation. A left-translation with a rotor R' is given by $\ell_{R'}R = R' *_C R$ and on the group vector manifold by $\ell_{R'}\boldsymbol{r}_R = \boldsymbol{r}_{R' *_C R}$. The left-translation induces a map $T_R\ell_{R'}$ between the tangent spaces at $\boldsymbol{r}_R$ and $\boldsymbol{r}_{R' *_C R}$. A vector field $\boldsymbol{a}(\boldsymbol{r}_R)$ on the group vector manifold is left invariant if $T_R\ell_{R'}\boldsymbol{a}(\boldsymbol{r}_R) = \boldsymbol{a}(\boldsymbol{r}_{R' *_C R})$. Left invariant vector fields on the group vector manifold can be obtained if one considers the multivector fields on the rotors given by $B_i^{\text{left}}(R) = R *_C \mathtt{B}_i$. For two rotors R and R' one has

$$B_i^{\text{left}}(R' *_C R) = R' *_C B_i^{\text{left}}(R). \tag{6.8}$$

Just as to each rotor R in the $\boldsymbol{\sigma}_a$-space corresponds a vector $\boldsymbol{r}_R$ in the $\boldsymbol{\tau}_a$-space there is also for each multivector field $B_i^{\text{left}}(R)$ in the $\boldsymbol{\sigma}_a$-space a left invariant vector field $\boldsymbol{\vartheta}_{B_i^{\text{left}}(R)}(\boldsymbol{r}_R) \equiv \boldsymbol{\vartheta}_i$ in the $\boldsymbol{\tau}_a$-space. These vector fields are closed under the Jacobi-Lie-bracket, i.e. they form a Lie subalgebra of all vector fields on $\boldsymbol{r}_R$ and they form a non-coordinate basis on $\boldsymbol{r}_R$, for the $SO(3)$-case one has for example $\boldsymbol{\vartheta}_i \cdot \boldsymbol{\vartheta}_j = \delta_{ij}$. The multivector fields $B_i^{\text{left}}(R)$ are uniquely defined by the bivectors at $R = 1$ and the corresponding left invariant vector fields are uniquely defined by their value in $\boldsymbol{r}_{R=1}$. In the $SO(3)$-example the tangent space at $\boldsymbol{r}_{R=1}(0,0,0) = \boldsymbol{\tau}_1$ is spanned by the vectors $\boldsymbol{\vartheta}_{\mathtt{B}_i} = \boldsymbol{\tau}_{i+1}$ and constitutes the $\mathfrak{so}(3)$ algebra in the $\boldsymbol{\tau}_a$-space, where the commutator product in the bivector algebra corresponds here in the $\mathfrak{so}(3)$-case to the vector cross product on the $\boldsymbol{\vartheta}_{\mathtt{B}_i}$-space, i.e.

$$\boldsymbol{\vartheta}_{\mathtt{B}_i \times \mathtt{B}_j} = -\boldsymbol{\vartheta}_{\mathtt{B}_i} \times \boldsymbol{\vartheta}_{\mathtt{B}_j}. \tag{6.9}$$

To each basis-bivector $\mathtt{B}_i$ of the bivector algebra a two-form Θ^i can be found so that $i_{\mathtt{B}_i}\Theta^j = \overline{\mathtt{B}_i} \cdot \Theta^j = \delta_i^j$ and to the two-forms Θ^i correspond then in the $\boldsymbol{\tau}_a$-space one-forms $\boldsymbol{\vartheta}^{\Theta^i} \equiv \boldsymbol{\vartheta}^i$ that generalize to reciprocal non-coordinate basis vector fields on $\boldsymbol{r}_R$, which clearly obey the Maurer-Cartan equation (4.70). For a r-form $\boldsymbol{A}^{(r)}$ on the group vector manifold that is vector-valued in the $\boldsymbol{\sigma}_a$-space one can then in analogy to (4.54) and with $i_{\boldsymbol{\vartheta}_1 \dots \boldsymbol{\vartheta}_r}\boldsymbol{A}^{(r)} = \boldsymbol{A}^{(r)}(\boldsymbol{\vartheta}_1 \dots \boldsymbol{\vartheta}_r)$ define the BRST-operator s as

$$\begin{aligned} \left(s\boldsymbol{A}^{(r)}\right)(\boldsymbol{\vartheta}_1\boldsymbol{\vartheta}_2 \dots \boldsymbol{\vartheta}_{r+1}) \quad = \quad & \sum_{n=1}^{r+1}(-1)^{n+1}\mathtt{B}_n \cdot \boldsymbol{A}^{(r)}\left(\boldsymbol{\vartheta}_1 \dots \check{\boldsymbol{\vartheta}}_n \dots \boldsymbol{\vartheta}_{r+1}\right) \\ & + \sum_{m<n}(-1)^{m+n}\boldsymbol{A}^{(r)}\left([\boldsymbol{\vartheta}_m, \boldsymbol{\vartheta}_n]_{JLB}\, \boldsymbol{\vartheta}_1 \dots \check{\boldsymbol{\vartheta}}_m \dots \check{\boldsymbol{\vartheta}}_n \dots \boldsymbol{\vartheta}_{r+1}\right). \end{aligned} \tag{6.10}$$

The s-operator can then be written as (see for example [20] and the references therein):

$$s = \mathtt{B}_i \cdot \otimes \boldsymbol{\vartheta}^i + \frac{1}{2}C_{ij}^k \boldsymbol{\vartheta}^j \boldsymbol{\vartheta}^i \frac{\partial}{\partial \boldsymbol{\vartheta}^k}. \tag{6.11}$$

The adjoint action of the rotor group on the bivector algebra is given by [5]

$$\mathrm{Ad}_R \mathtt{B} = R *_C \mathtt{B} *_C \overline{R}, \tag{6.12}$$

where $\mathtt{B} = b^i \mathtt{B}_i$ is a general element of the bivector algebra, to which in the $\boldsymbol{\vartheta}_{\mathtt{B}_i}$-space corresponds a vector $\boldsymbol{b} = b^i \boldsymbol{\vartheta}_{\mathtt{B}_i}$. Ad_R is a bivector algebra homomorphism, i.e. $\mathrm{Ad}_R(\mathtt{A} \times \mathtt{B}) = \mathrm{Ad}_R\mathtt{A} \times \mathrm{Ad}_R\mathtt{B}$ and a left action, i.e. $\mathrm{Ad}_{R *_C R'} = \mathrm{Ad}_R \mathrm{Ad}_{R'}$. For all elements R of the rotor group the adjoint action (6.12) constitutes the adjoint bivector orbit of $\mathtt{B}$, to which in the $\boldsymbol{\vartheta}_{\mathtt{B}_i}$-space corresponds an orbit vector manifold. In the $SO(3)$-case the adjoint action (6.12) leaves $|\mathtt{B}|^2 = \sum_{i=1}^{3}(b^i)^2 = |\boldsymbol{b}|^2$ invariant, so that the adjoint orbit vector manifold is an S^2.

Let now $\mathtt{A}$ be an element of the bivector algebra and consider the rotor $R(t) = e_{*_C}^{\frac{t}{2}\mathtt{A}}$. The adjoint action of this one-parameter rotor subgroup gives a curve in the bivector orbit, and the derivative at $t = 0$ is

$$\mathrm{ad}_{\mathtt{A}}\mathtt{B} = \left.\frac{d}{dt}\right|_{t=0} R(t) *_C \mathtt{B} *_C \overline{R(t)} = \mathtt{A} \times \mathtt{B}. \tag{6.13}$$

In the $\boldsymbol{\vartheta}_{\mathtt{B}_i}$-space the vector $\boldsymbol{\vartheta}_{\mathtt{A}\times\mathtt{B}}$ is the tangent vector in direction $\boldsymbol{\vartheta}_{\mathtt{A}}$ to the orbit vector manifold in the point $\boldsymbol{\vartheta}_{\mathtt{B}}$, i.e. $\boldsymbol{\vartheta}_{\mathtt{A}\times\mathtt{B}}$ generates the adjoint action corresponding to $\mathtt{A}$. It is also possible to define the coadjoint action Ad_R^* of the rotor group on a two-form Θ by

$$\overline{\mathtt{B}} \cdot \mathrm{Ad}_R^* \Theta = \overline{\mathrm{Ad}_R \mathtt{B}} \cdot \Theta, \tag{6.14}$$

which is the right action $\mathrm{Ad}^*_R\Theta = \overline{R} *_C \Theta *_C R$. The coadjoint left action is given by $\mathrm{Ad}^*_{\overline{R}}\Theta$. Infinitesimally one has $\overline{\mathtt{B}} \cdot \mathrm{ad}^*_{\mathtt{A}}\Theta = \overline{\mathrm{ad}_{\mathtt{A}}\mathtt{B}} \cdot \Theta$, or $\mathrm{ad}^*_{\mathtt{A}}\Theta = \Theta \times \mathtt{A}$. In the $SO(3)$-case the rotor acts on an euclidian space where the basis vectors and the reciprocal basis vectors are actually the same, so that $\mathtt{B}_i = \Theta^i$ and there is no difference between the adjoint and the coadjoint action.

In the above discussion the rotor R acts intrinsically from the left on a vector space. But more generally a rotor in an ambient space can also act from the left on a vector manifold $\boldsymbol{x}(x^i)$ by $\boldsymbol{x}' = R *_C \boldsymbol{x} *_C \overline{R}$ if $\boldsymbol{x}'$ is again a point in the vector manifold. The left-action of the rotor $R(t) = e_{*_C}^{\frac{t}{2}\mathtt{B}}$ induces on the vector manifold $\boldsymbol{x}(x^i)$ the vector field

$$\left.\frac{d}{dt}\right|_{t=0} R(t) *_C \boldsymbol{x} *_C \overline{R(t)} = \mathtt{B} \cdot \boldsymbol{x}. \tag{6.15}$$

Furthermore a short calculation shows that there is an algebra anti-homomorphism between the bivector algebra in the ambient space and the induced vector fields on the vector manifold, given by

$$[\mathtt{A} \cdot \boldsymbol{x}, \mathtt{B} \cdot \boldsymbol{x}]_{JLB} = -(\mathtt{A} \times \mathtt{B}) \cdot \boldsymbol{x}. \tag{6.16}$$

The rotor in the ambient space acts not only on the vectors $\boldsymbol{x}$ of the vector manifold, but in the same way also on tangent vectors $\boldsymbol{a}$ at the manifold which are vectors in the ambient space too. The transformation of $\boldsymbol{x}$ and $\boldsymbol{a}$ in the ambient space of the vector manifold induce a transformation in the tangent bundle. The tangent bundle manifold can be seen as a $2d$-dimensional vector manifold in a $(2d+2)$-dimensional ambient space with basis vectors $\boldsymbol{\sigma}_a$ and $\boldsymbol{\tau}_a$, i.e. as

$$(\boldsymbol{x} + \boldsymbol{a})(x^i, a^i) = x^a(x^i)\boldsymbol{\sigma}_a + a^j\xi^a_j(x^i)\boldsymbol{\tau}_a. \tag{6.17}$$

Analogously one can define multivector bundles, for example a bivector bundle manifold has the form

$$(\boldsymbol{x} + \mathtt{B})(x^i, B^{jk}) = x^a(x^i)\boldsymbol{\sigma}_a + B^{jk}\xi^a_j(x^i)\xi^b_k(x^i)\boldsymbol{\tau}_a\boldsymbol{\tau}_b. \tag{6.18}$$

The tangential lift of the rotor action is given by $R *_C \boldsymbol{x} *_C \overline{R} + R *_C \boldsymbol{a} *_C \overline{R}$, where the rotor acts on the $\boldsymbol{\tau}_a$-space in the same way as on the $\boldsymbol{\sigma}_a$-space. In the case of a flat vector manifold the tangent bundle is just a $2d$-dimensional vector space and the rotor acts separately and intrinsically on both subspaces. Instead of two rotors that act separately on the $\boldsymbol{\sigma}_a$ and $\boldsymbol{\tau}_a$ spaces one can consider also a lifted rotor with a bivector $\mathtt{B}_{\text{lifted}}$ that is the sum of the two single bivectors, so that one can write $R_{\text{lifted}} *_C (\boldsymbol{x} + \boldsymbol{a}) *_C \overline{R_{\text{lifted}}}$. If one describes the tangent vector in a reciprocal ambient space, i.e. as a one-form $\boldsymbol{\alpha}$ the cotangent bundle has the form

$$(\boldsymbol{x} + \boldsymbol{\alpha})(x^i, \alpha_i) = x^a(x^i)\boldsymbol{\sigma}_a + \alpha_i\xi^i_a(x^i)\boldsymbol{\tau}^a \tag{6.19}$$

and the corresponding cotangent lift is given by $\overline{R} *_C \boldsymbol{x} *_C R + \overline{R} *_C \boldsymbol{\alpha} *_C R$ or $\overline{R_{\text{lifted}}} *_C (\boldsymbol{x} + \boldsymbol{\alpha}) *_C R_{\text{lifted}}$.

In order to construct unitary transformations [21] one considers a $2n$-dimensional space with basis vectors $\boldsymbol{\alpha}_i$ and $\boldsymbol{\beta}_i$ for $i = 1, \ldots, n$ and a bivector

$$\mathtt{J} = \sum_{i=1}^{d} \boldsymbol{\alpha}_i\boldsymbol{\beta}_i = \sum_{i=1}^{d} \mathtt{J}_i. \tag{6.20}$$

The two subspaces spanned by $\boldsymbol{\alpha}_i$ and $\boldsymbol{\beta}_i$ should have the same metric, i.e. $\boldsymbol{\alpha}_i \cdot \boldsymbol{\alpha}_j = \boldsymbol{\beta}_i \cdot \boldsymbol{\beta}_j$ and $\boldsymbol{\alpha}_i \cdot \boldsymbol{\beta}_j = 0$. The $2n$-dimensional vector $\boldsymbol{x} = a^i\boldsymbol{\alpha}_i + b^i\boldsymbol{\beta}_i$ corresponds to an n-dimensional complex vector with components $x^k = \boldsymbol{x} \cdot \boldsymbol{\alpha}_k + \mathrm{i}\,\boldsymbol{x} \cdot \boldsymbol{\beta}_k = a^k + \mathrm{i}\,b^k$ and the complex internal product can be written as

$$\langle \boldsymbol{x} | \boldsymbol{y} \rangle = x^k\overline{y}_k = (\boldsymbol{x} \cdot \boldsymbol{\alpha}^k + \mathrm{i}\,\boldsymbol{x} \cdot \boldsymbol{\beta}^k)(\boldsymbol{y} \cdot \boldsymbol{\alpha}_k - \mathrm{i}\,\boldsymbol{y} \cdot \boldsymbol{\beta}_k) = \boldsymbol{x} \cdot \boldsymbol{y} + \mathrm{i}(\boldsymbol{x}\boldsymbol{y}) \cdot \mathtt{J}. \tag{6.21}$$

A unitary transformation generated by the rotor R leaves the above complex product invariant, i.e.

$$(\boldsymbol{x}\boldsymbol{y}) \cdot \mathtt{J} = \big((R *_C \boldsymbol{x} *_C \overline{R})(R *_C \boldsymbol{y} *_C \overline{R})\big) \cdot \mathtt{J} = (\boldsymbol{x}\boldsymbol{y}) \cdot (\overline{R} *_C \mathtt{J} *_C R), \tag{6.22}$$

which means that $\mathtt{J} = R *_C \mathtt{J} *_C \overline{R}$ is the defining relation for the unitary rotor. With the ansatz $R = e_{*_C}^{\mathtt{B}/2}$ one obtains the defining relation for the bivector $\mathtt{B}$

$$\mathtt{B} \times \mathtt{J} = 0, \tag{6.23}$$

which is solved by $\mathtt{B} = \boldsymbol{x}\boldsymbol{y} + (\boldsymbol{x} \cdot \mathtt{J})(\boldsymbol{y} \cdot \mathtt{J})$ Putting in this formula the basis vectors for $\boldsymbol{x}$ and $\boldsymbol{y}$ one obtains the n^2 basis bivectors of the $\mathfrak{u}(n)$-algebra:

$$\mathtt{E}_{ij} = \boldsymbol{\alpha}_i\boldsymbol{\alpha}_j + \boldsymbol{\beta}_i\boldsymbol{\beta}_j, \qquad \mathtt{F}_{ij} = \boldsymbol{\alpha}_i\boldsymbol{\beta}_j - \boldsymbol{\beta}_i\boldsymbol{\alpha}_j \qquad \text{and} \qquad \mathtt{J}_i = \boldsymbol{\alpha}_i\boldsymbol{\beta}_i \tag{6.24}$$

for $i < j = 1, \ldots, d$. It is easy to show that these basis bivectors form a closed algebra under the commutator product. The bivector $\mathtt{J}$ is part of the $\mathfrak{u}(n)$-algebra, if one excludes this generator of a global phase one obtains the $\mathfrak{su}(n)$-algebra.

In order to describe the $Gl(n)$ by rotors one proceeds in a way similar to the unitary case. One considers a $2n$-dimensional space spanned by the basis vectors $\boldsymbol{\alpha}_i$ and $\boldsymbol{\beta}_i$ for $i = 1, \ldots, n$, but now the metric in the spaces spanned by $\boldsymbol{\alpha}_i$ and $\boldsymbol{\beta}_i$ is opposite, i.e. the Clifford star product is given by

$$*_C = \exp\left[\eta_{ij}\frac{\overleftarrow{\partial}}{\partial\boldsymbol{\alpha}_i}\frac{\overrightarrow{\partial}}{\partial\boldsymbol{\alpha}_j} - \eta_{ij}\frac{\overleftarrow{\partial}}{\partial\boldsymbol{\beta}_i}\frac{\overrightarrow{\partial}}{\partial\boldsymbol{\beta}_j}\right]. \tag{6.25}$$

On this space a bivector $\mathtt{K} = \boldsymbol{\alpha}_i\boldsymbol{\beta}^i$ can be defined, so that one can decompose a vector $\boldsymbol{x}$ according to

$$\boldsymbol{x} = \frac{1}{2}\big(\boldsymbol{x} + \boldsymbol{x} \cdot \mathtt{K}\big) + \frac{1}{2}\big(\boldsymbol{x} - \boldsymbol{x} \cdot \mathtt{K}\big) = \boldsymbol{x}_+ + \boldsymbol{x}_-, \tag{6.26}$$

so that $\boldsymbol{x}_+ \cdot \boldsymbol{x}_+ = \boldsymbol{x}_- \cdot \boldsymbol{x}_- = 0$. There are then two subspaces V_+ and V_- defined by $\boldsymbol{x}_+ \cdot \mathtt{K} = \boldsymbol{x}_+$ and $\boldsymbol{x}_- \cdot \mathtt{K} = \boldsymbol{x}_-$. A $Gl(n)$-transformation transforms now a vector in V_+ into another vector in V_+, i.e.

$$(R *_C \boldsymbol{x}_+ *_C \overline{R}) \cdot \mathtt{K} = R *_C \boldsymbol{x}_+ *_C \overline{R}, \tag{6.27}$$

or $\mathtt{K} = R *_C \mathtt{K} *_C \overline{R}$. With the same argumentation as above one can see that a bivector generator must have the form $\mathtt{B} = \boldsymbol{x}\boldsymbol{y} - (\boldsymbol{x} \cdot \mathtt{K})(\boldsymbol{y} \cdot \mathtt{K})$, so that the n^2 basis bivectors of $\mathfrak{gl}(n)$ are

$$\mathtt{E}_{ij} = \boldsymbol{\alpha}_i\boldsymbol{\alpha}_j - \boldsymbol{\beta}_i\boldsymbol{\beta}_j, \qquad \mathtt{F}_{ij} = \boldsymbol{\alpha}_i\boldsymbol{\beta}_j - \boldsymbol{\beta}_i\boldsymbol{\alpha}_j \qquad \text{and} \qquad \mathtt{K}_i = \boldsymbol{\alpha}_i\boldsymbol{\beta}_i \tag{6.28}$$

for $i < j = 1, \ldots, n$.

The $Gl(n)$-case can be understood in another way if one transforms the variables of the vector $\boldsymbol{x} = a^i\boldsymbol{\alpha}_i + b^i\boldsymbol{\beta}_i$ into variables q^i, p^i, $\boldsymbol{\eta}_i$ and $\boldsymbol{\rho}_i$ according to

$$\begin{aligned} \boldsymbol{x}_+ &= \frac{1}{2}\big(\boldsymbol{x} + \boldsymbol{x} \cdot \mathtt{K}\big) = \frac{1}{2}(a^i - b^i)(\boldsymbol{\alpha}_i - \boldsymbol{\beta}_i) \equiv q^i\boldsymbol{\eta}_i && (6.29)\\ \boldsymbol{x}_- &= \frac{1}{2}\big(\boldsymbol{x} - \boldsymbol{x} \cdot \mathtt{K}\big) = \frac{1}{2}(a^i + b^i)(\boldsymbol{\alpha}_i + \boldsymbol{\beta}_i) \equiv p^i\boldsymbol{\rho}_i. && (6.30) \end{aligned}$$

It is then straight forward to transform the star product (6.25) and the generators (6.28) into these new variables. For the star product one obtains

$$*_C = \exp\left[\frac{\eta_{ij}}{2}\left(\frac{\overleftarrow{\partial}}{\partial\boldsymbol{\eta}_i}\frac{\overrightarrow{\partial}}{\partial\boldsymbol{\rho}_j}+\frac{\overleftarrow{\partial}}{\partial\boldsymbol{\rho}_i}\frac{\overrightarrow{\partial}}{\partial\boldsymbol{\eta}_j}\right)\right], \tag{6.31}$$

which is a fermionic version of the Moyal product

$$*_M = \exp\left[\frac{\mathrm{i}\hbar}{2}\eta^{ij}\left(\frac{\overleftarrow{\partial}}{\partial q^i}\frac{\overrightarrow{\partial}}{\partial p^j}-\frac{\overleftarrow{\partial}}{\partial p^i}\frac{\overrightarrow{\partial}}{\partial q^j}\right)\right]. \tag{6.32}$$

This suggests that the vector $\boldsymbol{x} = q^i\boldsymbol{\eta}_i + p^i\boldsymbol{\rho}_i$ can not only be transformed with a fermionic star exponential as described above, but can also be transformed in the bosonic coefficients with a bosonic star exponential according to [22]

$$e_{*_M}^{\alpha_{ij}M^{ij}} *_M q^k *_M e_{*_M}^{-\alpha_{ij}M^{ij}} = q^k + \alpha_{ij}\left[M^{ij}, q^k\right]_{*_M} + \frac{1}{2!}\alpha_{ij}\alpha_{lm}\left[M^{lm}, \left[M^{ij}, q^k\right]_{*_M}\right]_{*_M} + \ldots, \tag{6.33}$$

where $[f,g]_{*_M} = f *_M g - g *_M f$ is the star-commutator. In analogy to the fermionic case one can now demand that for a $Gl(n)$ transformation the q^k have to be a linear combination of the q^i alone and no terms in p^i should appear. This means that $\left[M^{ij}, q^k\right]_{*_M}$ must be a function of the q^i alone. This is achieved if one chooses the bosonic generators

$$E^{ij} = q^ip^j + q^jp^i, \qquad F^{ij} = q^ip^j - q^jp^i, \qquad \text{and} \qquad K^i = q^ip^i, \tag{6.34}$$

which form a closed algebra under the Moyal star-commutator.

7 Active and passive Rotations and the theoretical Prediction of Spin

A general multivector is now invariant under a combined transformation of the bosonic coefficients and a compensating transformation of the fermionic basis vectors. The bosonic transformation of the coefficients is an active transformation and the fermionic transformation of the basis vectors is a passive transformation. In a tuple formalism this difference cannot be made and so active and passive transformations are mixed up with left and right transformations, whereas in a multivector formalism one rather has that an active right transformation corresponds to a passive left transformation, and the other way round.

To illustrate the concept of active and passive transformations in the star product formalism one can consider rotations in space and space-time. In the three dimensional euclidian space with vectors $\boldsymbol{x} = x^i\boldsymbol{\sigma}_i$ the active rotations [22] are generated by the angular momentum functions

$$L^i = \varepsilon^{ijk}x^jp^k, \tag{7.1}$$

which fulfill with the three dimensional Moyal product

$$*_M = \exp\left[\frac{\mathrm{i}\hbar}{2}\sum_{i=1}^{3}\left(\frac{\overleftarrow{\partial}}{\partial x^i}\frac{\overrightarrow{\partial}}{\partial p^i}-\frac{\overleftarrow{\partial}}{\partial p^i}\frac{\overrightarrow{\partial}}{\partial x^i}\right)\right] \tag{7.2}$$

the active algebra

$$\left[L^i, L^j\right]_{*_M} = \mathrm{i}\hbar\varepsilon^{ijk}L^k. \tag{7.3}$$

An active left-rotation has then the form

$$\boldsymbol{x}' = \overline{U} *_M \boldsymbol{x} *_M U = e_{*_M}^{-\frac{\mathrm{i}}{\hbar}\alpha_k L^k} *_M \boldsymbol{x} *_M e_{*_M}^{\frac{\mathrm{i}}{\hbar}\alpha_k L^k} = \left(R^i_j x^j\right)\boldsymbol{\sigma}_i, \tag{7.4}$$

where the R^i_j is the well known rotation matrix. The corresponding passive rotation [21, 23] is generated by the bivectors

$$\mathtt{B}_i = \frac{1}{2}\varepsilon_{ijk}\boldsymbol{\sigma}_j\boldsymbol{\sigma}_k \tag{7.5}$$

that fulfill as seen above the passive algebra

$$\mathtt{B}_i \times \mathtt{B}_j = -\varepsilon_{ijk}\mathtt{B}_k, \tag{7.6}$$

so that the passive left-rotation is given by

$$\boldsymbol{x}' = \overline{R} *_C \boldsymbol{x} *_C R = e_{*_C}^{-\frac{1}{2}\alpha^k \mathtt{B}_k} *_C \boldsymbol{x} *_C e_{*_C}^{\frac{1}{2}\alpha^k \mathtt{B}_k} = x^i\left(R^j_i\boldsymbol{\sigma}_j\right). \tag{7.7}$$

It is clear that the above transformations generalize to arbitrary multivectors $A(x^i)$ and that such a multivector is invariant under a composed active and passive transformation [3]. The generator of such a composed transformation is then the sum of the active and passive generators, so that one has infinitesimally

$$\left[L^i + \frac{1}{2}\mathtt{B}_i, A(x^n)\right]_{*_{MC}} = \left[L^i, A(x^n)\right]_{*_M} + \mathtt{B}_i \times A(x^n) = \left[\varepsilon^{ijk}x^j\frac{\hbar}{\mathrm{i}}\frac{\partial}{\partial x^k} + \mathtt{B}_i\times\right]A(x^n). \tag{7.8}$$

In the conventional formalism one says that in quantum mechanics one has to go over from the angular momentum operator $\hat{L}_i$ to the operator $\hat{J}_i$ that includes also a Pauli matrix. In geometric algebra this follows from the invariance behavior of multivectors. Moreover the spin structure appears automatically if one deforms the minimal substituted Hamiltonian with the Moyal star product as shown in [7]. The star eigenfunctions of this Hamiltonian are then multivectors [9] that correspond to the Pauli spinors [24].

The same argumentation is better known from Dirac theory. A vector in the Minkowski space with basis vectors $\boldsymbol{\gamma}_\mu$ is given by $\boldsymbol{x} = x^\mu\boldsymbol{\gamma}_\mu$ and the active transformations can be done with a four dimensional Moyal star product

$$*_M = \exp\left[\frac{\mathrm{i}\hbar}{2}\eta^{\mu\nu}\left(\frac{\overleftarrow{\partial}}{\partial x^\mu}\frac{\overrightarrow{\partial}}{\partial p^\nu} - \frac{\overleftarrow{\partial}}{\partial p^\mu}\frac{\overrightarrow{\partial}}{\partial x^\nu}\right)\right], \tag{7.9}$$

where the nonstandard metric $\eta_{\mu\nu} = \mathrm{diag}(-1,1,1,1)$ should be chosen. The generators of an active Lorentz transformation are

$$M^{\mu\nu} = x^\mu p^\nu - p^\mu x^\nu, \tag{7.10}$$

where the generators of boosts and rotations are

$$K^i = M^{01} \qquad \text{and} \qquad L^i = \sum_{j<k}\varepsilon^{ijk}M^{jk}. \tag{7.11}$$

They form the following active Moyal star-commutator algebra

$$\left[L^i, L^j\right]_{*_M} = \mathrm{i}\hbar\varepsilon^{ijk}L^k, \qquad \left[L^i, K^j\right]_{*_M} = \mathrm{i}\hbar\varepsilon^{ijk}K^k \qquad \text{and} \qquad \left[K^i, K^j\right]_{*_M} = -\mathrm{i}\hbar\varepsilon^{ijk}L^k, \tag{7.12}$$

so that an active Lorentz transformation of the four-vector $\boldsymbol{x} = x^\mu \boldsymbol{\gamma}_\mu$ is given by

$$\boldsymbol{x}' = e_{*_M}^{-\frac{\mathrm{i}}{\hbar}\alpha_{\mu\nu}M^{\mu\nu}} *_M \boldsymbol{x} *_M e_{*_M}^{\frac{\mathrm{i}}{\hbar}\alpha_{\mu\nu}M^{\mu\nu}} = \left(\Lambda^\mu_\nu x^\nu\right)\boldsymbol{\gamma}_\mu, \tag{7.13}$$

where Λ^μ_ν is the well known Lorentz transformation matrix.

The corresponding passive Lorentz transformation is generated by the bivectors

$$\sigma_{\mu\nu} = \frac{I_{(4)}}{2} *_C \left[\boldsymbol{\gamma}_\mu, \boldsymbol{\gamma}_\nu\right]_{*_C}, \tag{7.14}$$

where $I_{(4)} = \boldsymbol{\gamma}_0\boldsymbol{\gamma}_1\boldsymbol{\gamma}_2\boldsymbol{\gamma}_3$ is the pseudoscalar. The generators for the passive boosts and rotations are

$$\mathtt{K}_i = \frac{1}{2}\sigma_{0i} \qquad \text{and} \qquad \mathtt{L}_i = \frac{1}{2}\sum_{j<k}\varepsilon_{ijk}\sigma_{jk} \tag{7.15}$$

and they satisfy in the case of the nonstandard metric (for the standard metric one has to replace i by -i in the active Lorentz algebra (7.12) and $I_{(4)}$ by $-I_{(4)}$ in the passive Lorentz algebra (7.16)):

$$\left[\mathtt{L}_i, \mathtt{L}_j\right]_{*_C} = -I_{(4)} *_C \varepsilon_{ijk}\mathtt{L}_k, \qquad \left[\mathtt{L}_i, \mathtt{K}_j\right]_{*_C} = -I_{(4)} *_C \varepsilon_{ijk}\mathtt{K}_k, \quad \text{and} \quad \left[\mathtt{K}_i, \mathtt{K}_j\right]_{*_C} = I_{(4)} *_C \varepsilon_{ijk}\mathtt{L}_k. \tag{7.16}$$

The passive Lorentz transformation is then given by

$$\boldsymbol{x}' = e_{*_C}^{\frac{1}{4}I_{(4)}*_C\alpha^{\mu\nu}\sigma_{\mu\nu}} *_C \boldsymbol{x} *_C e_{*_C}^{-\frac{1}{4}I_{(4)}*_C\alpha^{\mu\nu}\sigma_{\mu\nu}} = x^\mu\left(\Lambda^\nu_\mu\boldsymbol{\gamma}_\nu\right). \tag{7.17}$$

In Dirac theory the passive transformations are constructed a posteriori by demanding the invariance of the four-vector $p_\mu\boldsymbol{\gamma}^\mu$, just as the basis vectors of space-time are discovered a posteriori in a tuple notation by factorizing the Klein-Gordon equation.

8 Symplectic Vector Manifolds

A symplectic vector space can be considered as a $2d$-dimensional euclidian space with vectors

$$\boldsymbol{z} = z^a\boldsymbol{\zeta}_a = q^m\boldsymbol{\eta}_m + p^m\boldsymbol{\rho}_m, \tag{8.1}$$

where $a = 1, \ldots, 2d$ and $m = 1, \ldots, d$, and a closed two-form

$$\Omega = \frac{1}{2}\Omega_{ab}\boldsymbol{\zeta}^a\boldsymbol{\zeta}^b = \sum_{m=1}^{d}\boldsymbol{\eta}^m\boldsymbol{\rho}^m = \sum_{m=1}^{d}\boldsymbol{d}q^m\boldsymbol{d}p^m, \tag{8.2}$$

where Ω_{ab} is a non-degenerate, antisymmetric matrix [25]. The euclidian metric on the vector space defines a scalar product and a relation between vectors and one-forms. The two-form Ω gives now an additional possibility to establish such structures, i.e. one can define the symplectic scalar product as

$$\boldsymbol{z} \cdot_{Sy} \boldsymbol{w} \equiv \boldsymbol{i}_{\boldsymbol{z}\boldsymbol{w}}\Omega = (\boldsymbol{w}\boldsymbol{z}) \cdot \Omega = \boldsymbol{z} \cdot (\Omega \cdot \boldsymbol{w}) = z^a\Omega_{ab}w^b \tag{8.3}$$

and furthermore one can map with Ω a vector into a one-form according to $\boldsymbol{z}^\flat = i_{\boldsymbol{z}}\Omega = \boldsymbol{z}\cdot\Omega$ (the other possibility used in [12] is $\Omega\cdot\boldsymbol{z} = -\boldsymbol{z}\cdot\Omega$). The inverse map of a one-form into a vector can be described with the bivector

$$\mathtt{J} = \frac{1}{2}J^{ab}\boldsymbol{\zeta}_a\boldsymbol{\zeta}_b = \frac{1}{2}\sum_{a,b=1}^{2d}\Omega_{ab}\boldsymbol{\zeta}_a\boldsymbol{\zeta}_b = \sum_{m=1}^{d}\boldsymbol{\eta}_m\boldsymbol{\rho}_m, \tag{8.4}$$

so that the vector corresponding to a one-form $\boldsymbol{\omega}$ is given by $\boldsymbol{\omega}^\natural = \mathtt{J}\cdot\boldsymbol{\omega}$. The map $\natural$ should be inverse to $\flat$, from which $J^{ab} = (\Omega^{-1}_{ab})^T = \Omega^{ba}$ follows. Especially with the nabla operator $\boldsymbol{\nabla} = \boldsymbol{d} = \boldsymbol{\zeta}^a\partial_a$ and $\boldsymbol{d}^\natural = \mathtt{J}\cdot\boldsymbol{d}$ the Hamilton equations can be written as in [12]:

$$\dot{\boldsymbol{z}} = \boldsymbol{d}^\natural H. \tag{8.5}$$

Furthermore the Poisson bracket can be written as

$$\{F,G\}_{PB} = F\overleftarrow{\boldsymbol{d}}\cdot_{Sy}\overrightarrow{\boldsymbol{d}}\,G = J^{ab}\frac{\partial F}{\partial x^a}\frac{\partial G}{\partial x^b}. \tag{8.6}$$

The bivector $\mathtt{J}$ plays the role of the compatible complex structure [25], because

$$(\boldsymbol{z}\cdot\mathtt{J})\cdot_{Sy}(\boldsymbol{w}\cdot\mathtt{J}) = \boldsymbol{z}\cdot_{Sy}\boldsymbol{w} \qquad\text{and}\qquad \boldsymbol{z}\cdot_{Sy}(\boldsymbol{z}\cdot\mathtt{J}) > 0 \qquad \forall\boldsymbol{z}\neq 0. \tag{8.7}$$

Furthermore one has $\mathtt{J}\cdot\mathtt{J} = -1$, $(\boldsymbol{z}\cdot\mathtt{J})\cdot\mathtt{J} = -\boldsymbol{z}$ and the symplectic scalar product can be written as $\boldsymbol{z}\cdot_{Sy}\boldsymbol{w} = (\boldsymbol{z}\cdot\mathtt{J})\cdot\boldsymbol{w}$. A metric space with a two-form Ω and a compatible complex structure is a Kähler space.

A symplectic vector manifold is an even-dimensional vector manifold with a closed two-form $\Omega(\boldsymbol{x}) = \frac{1}{2}\Omega_{ij}\boldsymbol{\xi}^i\boldsymbol{\xi}^j$, i.e. with $\partial_i\Omega_{jk} + \partial_j\Omega_{ki} + \partial_k\Omega_{ij} = 0$. The tangent spaces at the symplectic vector manifold are symplectic vector spaces. A vector field $\boldsymbol{z}(\boldsymbol{x})$ on a symplectic vector manifold is symplectic if $\boldsymbol{z}^\flat$ is closed, i.e. if $\boldsymbol{d}(\boldsymbol{z}\cdot\Omega) = 0$. Symplectic vector fields conserve the symplectic structure, i.e. $\mathscr{L}_{\boldsymbol{z}}\Omega = \boldsymbol{d}i_{\boldsymbol{z}}\Omega = 0$ and they form an algebra under the Jacobi-Lie bracket, i.e. for two symplectic vector fields $\boldsymbol{z}(\boldsymbol{x})$ and $\boldsymbol{w}(\boldsymbol{x})$ one has $\boldsymbol{d}\big([\boldsymbol{z},\boldsymbol{w}]_{JLB}\cdot\Omega\big) = 0$. If $\boldsymbol{z}^\flat$ is not only closed, but is also exact the vector field is called hamiltonian. According to the Poincaré lemma every closed form is locally exact, so that a symplectic vector field is locally hamiltonian. This means for a local hamiltonian vector field $\boldsymbol{h}_H$ exists locally a function H so that

$$\boldsymbol{h}_H\cdot\Omega = \boldsymbol{d}H. \tag{8.8}$$

In the coordinate basis the hamiltonian vector field reads

$$\boldsymbol{h}_H = \boldsymbol{d}^\natural H = J^{ij}(\partial_j H)\boldsymbol{\xi}_i. \tag{8.9}$$

With a hamiltonian vector field the Poisson bracket can then be written as

$$\mathscr{L}_{\boldsymbol{h}_H}F = \boldsymbol{h}_H\cdot\boldsymbol{d}F = \{F,H\}_{PB}, \tag{8.10}$$

or, using (8.8) in this equation, as

$$\{F,G\}_{PB} = i_{\boldsymbol{h}_F\boldsymbol{h}_G}\Omega = (\boldsymbol{h}_G\boldsymbol{h}_F)\cdot\Omega. \tag{8.11}$$

It is easy to see that the hamiltonian vector fields form a Lie subalgebra of the symplectic vector fields with

$$[\boldsymbol{h}_F,\boldsymbol{h}_G]_{JLB} = -\boldsymbol{h}_{\{F,G\}_{PB}}. \tag{8.12}$$

Given a symplectic vector field $\boldsymbol{z}$ that preserves the Hamilton function H, i.e. $\mathscr{L}_{\boldsymbol{z}}\Omega = \Omega$ and $\mathscr{L}_{\boldsymbol{z}}H = 0$, this symplectic vector field $\boldsymbol{z}$ can be written locally as a hamiltonian vector field $\boldsymbol{h}_F$ with

$$\mathscr{L}_{\boldsymbol{h}_F} H = \boldsymbol{h}_F \cdot \boldsymbol{d}H = \{F, H\}_{PB} = 0, \tag{8.13}$$

which shows that F is a conserved quantity. This is Noethers theorem for the symplectic case.

The metric $g_{ij}(\boldsymbol{x})$ on the vector manifold is induced by the ambient space and exists naturally on the vector manifold. It was used in the above discussion just to contract vector fields and forms with the scalar product. But this contraction is actually independent of the metric. The metric can be used to define a compatible almost complex structure. This is here a bivector field $\mathtt{J}(\boldsymbol{x})$, that maps via the scalar product a tangent vector into another tangent vector. If the structures $g_{ij}(\boldsymbol{x})$, $\mathtt{J}(\boldsymbol{x})$ and $\Omega(\boldsymbol{x})$ are compatible the metric scalar product of two tangent vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ in a point $\boldsymbol{x}$ can be written as $\boldsymbol{a} \cdot \boldsymbol{b} = \boldsymbol{a} \cdot_{Sy} (\boldsymbol{b} \cdot \mathtt{J})$ and the symplectic product can be written as $\boldsymbol{a} \cdot_{Sy} \boldsymbol{b} = (\boldsymbol{a} \cdot \mathtt{J}) \cdot \boldsymbol{b}$. A symplectic vector manifold with these three compatible structures corresponds to a Kähler vector manifold (if the Nijenhuis torsion vanishes).

Symplectic manifolds of special physical interest are cotangent bundles, for which the symplectic two-form is globally exact. The cotangent bundle of a d-dimensional euclidian vector space is a $2d$-dimensional euclidian vector space with elements $\boldsymbol{q} + \boldsymbol{\pi} = q^m \boldsymbol{\eta}_m + p_m \boldsymbol{\rho}^m$. On this vector space one can define with a vector $\boldsymbol{a} + \boldsymbol{\omega} = a^m \boldsymbol{\eta}_m + \omega_m \boldsymbol{\rho}^m$ a canonical one-form $\boldsymbol{\theta}(\boldsymbol{q} + \boldsymbol{\pi})$ by

$$(\boldsymbol{a} + \boldsymbol{\omega}) \cdot \boldsymbol{\theta}(\boldsymbol{q} + \boldsymbol{\pi}) = a^m p_m, \tag{8.14}$$

so that $\boldsymbol{\theta} = p_m \boldsymbol{\eta}^m = p_m \boldsymbol{d}q^m$, where the nabla operator is given by $\boldsymbol{\nabla} = \boldsymbol{d} = \boldsymbol{\eta}^m \frac{\partial}{\partial q^m} + \boldsymbol{\rho}_m \frac{\partial}{\partial p_m}$. The symplectic two-form on the cotangent bundle can then be obtained as

$$\Omega = -\boldsymbol{d\theta} = \boldsymbol{\eta}^m \boldsymbol{\rho}_m = \boldsymbol{d}q^m \boldsymbol{d}p_m. \tag{8.15}$$

The above definitions generalize readily to the case of a cotangent bundle of a d-dimensional vector manifold. In a $(2d+2)$-dimensional ambient vector space with basis vectors $\boldsymbol{\sigma}_a$ and $\boldsymbol{\tau}^a$ the cotangent bundle can be described as the $2d$-dimensional vector manifold $(\boldsymbol{q}+\boldsymbol{\pi})(q^i, p_i) = q^a(q^i)\boldsymbol{\sigma}_a + p_j \xi^j_a(q^i)\boldsymbol{\tau}^a$, with tangent vectors $\boldsymbol{a} + \boldsymbol{\omega} = a^i \xi^a_i \boldsymbol{\sigma}_a + \omega_i \xi^i_a \boldsymbol{\tau}^a$. With a projection operator $T\pi_{\boldsymbol{q}}$ defined as

$$T\pi_{\boldsymbol{q}}(\boldsymbol{a} + \boldsymbol{\omega}) = T\pi_{\boldsymbol{q}}(\boldsymbol{a}) = a^i \xi^a_i \boldsymbol{\tau}_a, \tag{8.16}$$

one can write (8.14) as $(\boldsymbol{a} + \boldsymbol{\omega}) \cdot \boldsymbol{\theta}(\boldsymbol{q} + \boldsymbol{\pi}) = T\pi_{\boldsymbol{q}}(\boldsymbol{a} + \boldsymbol{\omega}) \cdot \boldsymbol{\pi}$, so that $\boldsymbol{\theta} = p_i \boldsymbol{\xi}^i = p_i \boldsymbol{d}q^i$.

In the discussion so far the symplectic structure was defined via a two-form and a compatible metric star product, which led to a Kähler vector manifold. But a metric structure is actually not necessary to define a symplectic structure. In the case of a cotangent bundle it suffices to use the natural duality on this space. This duality can also be described with a star product, for example on the cotangent bundle of a vector space one can define

$$F *_D G = F \exp\left[\frac{\overleftarrow{\partial}}{\partial \boldsymbol{\eta}_a} \frac{\overrightarrow{\partial}}{\partial \boldsymbol{\rho}^a}\right] G, \tag{8.17}$$

so that (8.14) reads $i_{\boldsymbol{a}+\boldsymbol{\omega}} \boldsymbol{\theta}(\boldsymbol{q} + \boldsymbol{\pi}) = i_{\boldsymbol{a}} \boldsymbol{\pi} = \langle \boldsymbol{a} *_D \boldsymbol{\pi} \rangle_0 = \boldsymbol{a} \cdot_D \boldsymbol{\pi}$ and further $i_{(\boldsymbol{a}+\boldsymbol{\omega})(\boldsymbol{b}+\boldsymbol{\chi})}\Omega = \boldsymbol{a} \cdot_D \boldsymbol{\chi} - \boldsymbol{b} \cdot_D \boldsymbol{\omega}$, which can easily be generalized to manifolds [26]. The other possibility is to define a symplectic star product, by using Ω_{ij} instead of the metric η_{ij} in the fermionic star product. On a $2d$-dimensional vector space the symplectic star product in Darboux coordinates is given by

$$F *_{Sy} G = F \exp\left[\Omega_{ab} \frac{\overleftarrow{\partial}}{\partial \boldsymbol{\zeta}_a} \frac{\overrightarrow{\partial}}{\partial \boldsymbol{\zeta}_b}\right] G = F \exp\left[\sum_{m=1}^{d} \left(\frac{\overleftarrow{\partial}}{\partial \boldsymbol{\eta}_m} \frac{\overrightarrow{\partial}}{\partial \boldsymbol{\rho}_m} - \frac{\overleftarrow{\partial}}{\partial \boldsymbol{\rho}_m} \frac{\overrightarrow{\partial}}{\partial \boldsymbol{\eta}_m} \right)\right] G. \tag{8.18}$$

On a $2d$-dimensional vector manifold the tangent space can also be spanned by Darboux basis vectors $\boldsymbol{\eta}_m = \eta^i_m \boldsymbol{\xi}_i$ and $\boldsymbol{\rho}_m = \rho^i_m \boldsymbol{\xi}_i$ so that one has analogously

$$F *_{Sy} G = F \exp\left[\Omega_{ij} \frac{\overleftarrow{\partial}}{\partial \boldsymbol{\xi}_i} \frac{\overrightarrow{\partial}}{\partial \boldsymbol{\xi}_j}\right] G = F \exp\left[\sum_{m=1}^{d} \left(\frac{\overleftarrow{\partial}}{\partial \boldsymbol{\eta}_m} \frac{\overrightarrow{\partial}}{\partial \boldsymbol{\rho}_m} - \frac{\overleftarrow{\partial}}{\partial \boldsymbol{\rho}_m} \frac{\overrightarrow{\partial}}{\partial \boldsymbol{\eta}_m} \right)\right] G. \tag{8.19}$$

The indices are now lowered and raised with Ω_{ij}, i.e. for a tangent vector $\boldsymbol{a} = a^i \boldsymbol{\xi}_i$ one has $a_i = \Omega_{ij} a^j$ and $\boldsymbol{\xi}^i = \Omega^{ij} \boldsymbol{\xi}_j$, where $\Omega_{ij} \Omega^{jk} = \delta^k_i$. The relations $\flat$ and $\natural$ between vectors and one-forms can then be written as $\boldsymbol{a}^\flat = a^i \Omega_{ij} \boldsymbol{\xi}^j = (\Omega^T_{ji} a^i) \boldsymbol{\xi}^j$ and $\boldsymbol{\omega}^\natural = \omega_i \Omega^{ij} \boldsymbol{\xi}_j = (\Omega^{ji\,T} \omega_i) \boldsymbol{\xi}_j = J^{ji} \omega_i \boldsymbol{\xi}_j$. Furthermore it follows for the scalar products

$$\boldsymbol{\xi}_i \cdot_{Sy} \boldsymbol{\xi}_j = \Omega_{ij}, \qquad \boldsymbol{\xi}^i \cdot_{Sy} \boldsymbol{\xi}_j = -\boldsymbol{\xi}_j \cdot_{Sy} \boldsymbol{\xi}^i = \delta^i_j \qquad \text{and} \qquad \boldsymbol{\xi}^i \cdot_{Sy} \boldsymbol{\xi}^j = -\Omega^{ij} = J^{ij}. \tag{8.20}$$

If one establishes the symplectic structure with the symplectic star product and not with a metric star product and a two-form, the contraction of vectors and one-forms has to be defined with the symplectic scalar product $\boldsymbol{\xi}_i \cdot_{Sy} \boldsymbol{\xi}^j = -\delta^j_i$. This leads to a different sign structure compared with the case of a metric star product, for example instead of (8.8) one has for a hamiltonian vector field on a vector space with a symplectic star product $\boldsymbol{h}_H \cdot_{Sy} \Omega = -\boldsymbol{d}H$ and since $\boldsymbol{a} \cdot_{Sy} \boldsymbol{\partial} = -\boldsymbol{a} \cdot \boldsymbol{\partial}$ there is no minus sign on the right side of (8.12). So these two sign conventions correspond to the use of a metric or a symplectic star product on the vector space.

9 Active and passive Transformations on the Phase Space

A flat phase space can be considered as an $2d$-dimensional euclidian vector space with vectors (8.1) and a two-form (8.2). The time development is described by the hamiltonian vector field $\boldsymbol{h}_H = \dot{q}^n \boldsymbol{\eta}_n + \dot{p}^n \boldsymbol{\rho}_n = J^{ij} \partial_j H \boldsymbol{\zeta}_i$, so that one has for a scalar phase space function f

$$\dot{f} = \dot{\boldsymbol{z}} \cdot (\boldsymbol{d}f) = (\boldsymbol{h}_H \cdot \boldsymbol{d}) f = \mathscr{L}_{\boldsymbol{h}_H} f = \{f, H\}_{PB}. \tag{9.1}$$

where $\boldsymbol{h}_H \cdot \boldsymbol{d}$ is the Liouville operator. The above equation for the time development can immediately be generalized from 0-forms f to arbitrary r-forms. For example the time development of the symplectic two-form is given by $\dot{\Omega} = \mathscr{L}_{\boldsymbol{h}_H} \Omega = 0$, which means that the symplectic form is preserved by the time evolution.

The temporal development of a system can be described by an active time transformation of the coefficients, which corresponds to the Hamilton equations

$$\dot{z}^i = \mathscr{L}_{\boldsymbol{h}_H} z^i = J^{ij} \partial_j H. \tag{9.2}$$

In the formalism of geometric algebra it is also possible to write down a time transformation of the basis vectors

$$\dot{\boldsymbol{\zeta}}_i = \mathscr{L}_{\boldsymbol{h}_H} \boldsymbol{\zeta}_i = -J^{jk} \partial_k \partial_i H \boldsymbol{\zeta}_j, \tag{9.3}$$

which corresponds to the Jacobi equation that appeared in the path integral formulation of classical mechanics [13].

Active and passive time development can directly be discussed for the example of the harmonic oscillator. The Hamiltonian $H = \frac{1}{2}(p^2 + q^2)$ generates via the star exponential $U(t) = e_{*_M}^{-\frac{i}{\hbar}Ht}$ an active rotation of the state vector $\boldsymbol{z}_0 = q\boldsymbol{\eta} + p\boldsymbol{\rho}$ according to [27]

$$\boldsymbol{z}(t) = \overline{U(t)} *_M \boldsymbol{z}_0 *_M U(t) = (q\cos t + p\sin t)\boldsymbol{\eta} + (-q\sin t + p\cos t)\boldsymbol{\rho} = q(t)\boldsymbol{\eta} + p(t)\boldsymbol{\rho}. \tag{9.4}$$

The same transformation passively can be achieved with the rotor $R(t) = e_{*_C}^{\frac{1}{2}\mathtt{H}t}$ and the bivector $\mathtt{H} = \boldsymbol{\eta}\boldsymbol{\rho}$ as

$$\boldsymbol{z}(t) = R(t) *_C \boldsymbol{z}_0 *_C \overline{R(t)} = q(\cos t\,\boldsymbol{\eta} - \sin t\,\boldsymbol{\rho}) + p(\sin t\,\boldsymbol{\eta} + \cos t\,\boldsymbol{\rho}) = q\boldsymbol{\eta}(t) + p\boldsymbol{\rho}(t). \tag{9.5}$$

With the hamiltonian vector-field $\boldsymbol{h}_H = p\boldsymbol{\eta} - q\boldsymbol{\rho}$ and the relation $\{f,g\}_{PB} = \lim_{\hbar\to 0}\frac{1}{\mathrm{i}\hbar}\,[f,g]_{*_M}$ the active Hamilton equations $\dot{z}^i = \mathscr{L}_{\boldsymbol{h}_H} z^i$ can be written as

$$\dot{q} = \lim_{\hbar\to 0}\frac{1}{\mathrm{i}\hbar}\,[q,H]_{*_M} = p \qquad \text{and} \qquad \dot{p} = \lim_{\hbar\to 0}\frac{1}{\mathrm{i}\hbar}\,[p,H]_{*_M} = -q. \tag{9.6}$$

With (9.3) one can then calculate the corresponding time inverted passive Hamilton equations. Using the Clifford star commutator defined by

$$\left[A_{(r)}, B_{(s)}\right]_{*_C} = A_{(r)} *_C B_{(s)} - (-1)^{rs} B_{(s)} *_C A_{(r)} \tag{9.7}$$

these equations can be written as

$$\dot{\boldsymbol{\eta}} = \frac{1}{\mathrm{i}}\,[\boldsymbol{\eta},\mathtt{H}]_{*_C} = \boldsymbol{\rho} \qquad \text{and} \qquad \dot{\boldsymbol{\rho}} = \frac{1}{\mathrm{i}}\,[\boldsymbol{\rho},\mathtt{H}]_{*_C} = -\boldsymbol{\eta}, \tag{9.8}$$

where $\mathtt{H} = \frac{\mathrm{i}}{2}\boldsymbol{\eta}\boldsymbol{\rho}$ is the passive Hamiltonian. The passive Hamiltonian is connected with the active one over (9.7) and (9.3) by

$$\frac{1}{\mathrm{i}}\,[\boldsymbol{\zeta}_i,\mathtt{H}]_{*_C} = -J^{jk}\partial_k\partial_i H\boldsymbol{\zeta}_j. \tag{9.9}$$

The passive Hamiltonian $\mathtt{H}$ is here just the free Hamiltonian of pseudoclassical mechanics [28] (the additional factor $\frac{1}{2}$ is due to the definition of the Clifford product which is defined without a factor $\frac{1}{2}$, see for example [7]).

A Lagrangian that takes into account both the time development according to (9.2) and the time development according to (9.3) should be called the extended Lagrangian and has the form

$$\begin{aligned} \widetilde{\mathcal{L}}_E &= y_i\left(\dot{z}^i - J^{ij}\partial_j H\right) + \mathrm{i}\boldsymbol{\zeta}_j\left(\partial_t\delta^j_l - J^{jk}\partial_l\partial_k H\right)\boldsymbol{\lambda}^l \\ &= y_i\dot{z}^i + \mathrm{i}\boldsymbol{\zeta}_j\dot{\boldsymbol{\lambda}}^j - \widetilde{\mathcal{H}}_E, \end{aligned} \tag{9.10}$$

where the extended Hamiltonian $\widetilde{\mathcal{H}}_E$ is given by

$$\widetilde{\mathcal{H}}_E = y_i J^{ij}\partial_j H + \mathrm{i}\boldsymbol{\zeta}_j J^{jk}\partial_l\partial_k H\boldsymbol{\lambda}^l. \tag{9.11}$$

The extended Lagrangian first appeared in the path integral approach to classical mechanics [13, 14], where the classical analogue of the quantum generating functional was considered:

$$Z_{CM}[J] = N\int Dz\,\delta\left[z(t) - z_{cl}(t)\right]\,\exp\left[\int dt\,J\phi\right]. \tag{9.12}$$

The delta function here constrains all possible trajectories to the classical trajectory obeying (9.2). It can be written as

$$\delta\left[z(t)-z_{cl}(t)\right]=\delta\left[\dot{z}^i-\Omega^{ij}\partial_j H\right]\det\left[\delta^i_j\partial_t-\Omega^{ik}\partial_k\partial_j H\right]. \tag{9.13}$$

The delta function on the right side can be expressed by a Fourier transform

$$\delta\left[\dot{z}^i-\Omega^{ij}\partial_j H\right]=\int Dy_i \exp\left[\mathrm{i}\int dt\, y_i\left(\dot{z}^i-\Omega^{ij}\partial_j H\right)\right] \tag{9.14}$$

and the determinant can be written in terms of Grassmann variables as

$$\det\left[\delta^i_j\partial_t-\Omega^{ik}\partial_k\partial_j H\right]=\int D\boldsymbol{\lambda}^i D\boldsymbol{\zeta}_i\, \exp\left[-\int dt\, \boldsymbol{\zeta}_i\left[\delta^i_j\partial_t-\Omega^{ik}\partial_k\partial_j H\right]\boldsymbol{\lambda}^j\right], \tag{9.15}$$

so that $Z_{CM}[0]$ becomes

$$Z_{CM}[0]=\int Dz^i Dy_i D\boldsymbol{\lambda}^j D\boldsymbol{\zeta}_j\, \exp\left[\mathrm{i}\int dt\, \widetilde{\mathcal{L}}_E\right]. \tag{9.16}$$

The important point is here, that the path integral formalism of classical mechanics gives the fermionic basis vectors of geometric algebra the physical interpretation of ghosts. On the other hand the superanalytic formulation of geometric algebra has naturally the fermionic structures that in the conventional formalism have to be added ad hoc and per hand.

The z^i and $\boldsymbol{\zeta}_i$ form together with the newly introduced variables y_i and $\boldsymbol{\lambda}^i$ the extended phase space. On this extended phase space one can then introduce an extended canonical structure. This can easily be done in analogy to the Moyal and the Clifford star product structures of the phase space. Defining the extended Moyal-Clifford star product as

$$F *_{EMC} G = F\, \exp\left[\frac{\mathrm{i}}{2}\left(\frac{\overleftarrow{\partial}}{\partial z^k}\frac{\overrightarrow{\partial}}{\partial y_k}-\frac{\overleftarrow{\partial}}{\partial y_k}\frac{\overrightarrow{\partial}}{\partial z^k}\right)+\frac{1}{2}\left(\frac{\overleftarrow{\partial}}{\partial \boldsymbol{\lambda}^k}\frac{\overrightarrow{\partial}}{\partial \boldsymbol{\zeta}_k}+\frac{\overleftarrow{\partial}}{\partial \boldsymbol{\zeta}_k}\frac{\overrightarrow{\partial}}{\partial \boldsymbol{\lambda}^k}\right)\right] G \tag{9.17}$$

the extended Poisson bracket has the form

$$\{F,G\}_{EPB}=\frac{1}{\mathrm{i}}\left[F *_{EMC} G-(-1)^{\epsilon(F)\epsilon(G)} G *_{EMC} F\right], \tag{9.18}$$

where $\epsilon(F)$ gives the Grassmann grade of F. In the bosonic part of the extended Clifford star product a factor $\hbar$ can be included like in the Moyal product, so that in the definition of the extended Poisson bracket (9.18) the limit $\hbar\rightarrow 0$ has to be taken. The extended canonical relations are then given by

$$\{z^i, y_j\}_{EPB}=\delta^i_j \qquad \text{and} \qquad \{\boldsymbol{\zeta}_i, \boldsymbol{\lambda}^j\}_{EPB}=-\mathrm{i}\delta^j_i, \tag{9.19}$$

while all other extended Poisson brackets vanish. Furthermore one can calculate the equations of motion as

$$\dot{z}^i = \{z^i, \widetilde{\mathcal{H}}_E\}_{EPB}=\Omega^{ij}\partial_j H, \tag{9.20}$$
$$\dot{\boldsymbol{\zeta}}_i = \{\boldsymbol{\zeta}_i, \widetilde{\mathcal{H}}_E\}_{EPB}=-\Omega^{jk}\partial_k\partial_i H\boldsymbol{\zeta}_j, \tag{9.21}$$
$$\dot{y}_i = \{y_i, \widetilde{\mathcal{H}}_E\}_{EPB}=-z_j\Omega^{jk}\partial_k\partial_i H-\mathrm{i}\boldsymbol{\zeta}_j\Omega^{jk}\partial_k\partial_l\partial_i H\boldsymbol{\lambda}^l, \tag{9.22}$$
$$\dot{\boldsymbol{\lambda}}^i = \{\boldsymbol{\lambda}^i, \widetilde{\mathcal{H}}_E\}_{EPB}=\Omega^{ij}\partial_j\partial_k H\boldsymbol{\lambda}^k. \tag{9.23}$$

The extended Hamiltonian also generates the time development of r-vectors and r-forms according to [17]

$$\dot{X} = \mathscr{L}_{\boldsymbol{h}} X = \{X, \widetilde{\mathcal{H}}_E\}_{EPB}. \tag{9.24}$$

Having now a superanalytic formalism for classical mechanics that takes into account active and passive time development, one can ask if there is a supersymmetry in this formalism, i.e. a symmetry that relates the bosonic coefficients with the fermionic basis vectors. This supersymmetry was found by Gozzi et al. in [13]. There was shown that $\widetilde{\mathcal{H}}_E$ is invariant under the following BRST-transformation

$$\delta z^k = \boldsymbol{\varepsilon}\boldsymbol{\lambda}^k, \qquad \delta\boldsymbol{\zeta}_k = \mathrm{i}\boldsymbol{\varepsilon} y_k, \qquad \delta\boldsymbol{\lambda}^k = \delta y_k = 0 \tag{9.25}$$

and the following anti-BRST-transformation

$$\delta z^k = -\boldsymbol{\varepsilon}\Omega^{kl}\boldsymbol{\zeta}_l, \qquad \delta\boldsymbol{\lambda}^k = \mathrm{i}\overline{\boldsymbol{\varepsilon}}\Omega^{kl} y_l, \qquad \delta\boldsymbol{\zeta}_k = \delta y_k = 0, \tag{9.26}$$

where $\boldsymbol{\varepsilon}$ and $\overline{\boldsymbol{\varepsilon}}$ are Grassmann variables. These symmetries are generated by

$$\boldsymbol{Q}_{BRST} = y_j \boldsymbol{\lambda}^j \qquad \text{and} \qquad \boldsymbol{Q}_{\overline{BRST}} = \boldsymbol{\zeta}_j \Omega^{jk} y_k \tag{9.27}$$

according to $\delta X = \{X, \boldsymbol{\varepsilon}\boldsymbol{Q}_{BRST} + \overline{\boldsymbol{\varepsilon}}\boldsymbol{Q}_{\overline{BRST}}\}_{EPB}$. The two charges $\boldsymbol{Q}_{BRST}$ and $\boldsymbol{Q}_{\overline{BRST}}$ are conserved, i.e.

$$\{\boldsymbol{Q}_{BRST}, \widetilde{\mathcal{H}}_E\}_{EPB} = \{\boldsymbol{Q}_{\overline{BRST}}, \widetilde{\mathcal{H}}_E\}_{EPB} = 0 \tag{9.28}$$

and fulfill

$$\{\boldsymbol{Q}_{BRST}, \boldsymbol{Q}_{BRST}\}_{EPB} = \{\boldsymbol{Q}_{\overline{BRST}}, \boldsymbol{Q}_{\overline{BRST}}\}_{EPB} = \{\boldsymbol{Q}_{BRST}, \boldsymbol{Q}_{\overline{BRST}}\}_{EPB} = 0. \tag{9.29}$$

10 Poisson Vector Manifolds

A vector manifold M with a bivector $\mathtt{J}(\boldsymbol{x}) = \frac{1}{2} J^{ij}\boldsymbol{\xi}_i\boldsymbol{\xi}_j$ and

$$J^{ij}\partial_i J^{kl} + J^{ik}\partial_i J^{lj} + J^{il}\partial_i J^{jk} = 0 \tag{10.1}$$

is a Poisson vector manifold, where (10.1) can be written with (3.24) as $[\mathtt{J},\mathtt{J}]_{SNB} = 0$. The bivector $\mathtt{J}$ defines as discussed above a map from $T^*_{\boldsymbol{x}}M$ to $T_{\boldsymbol{x}}M$ by $\boldsymbol{\alpha}^\natural = \mathtt{J}\cdot\boldsymbol{\alpha} = J^{ij}\alpha_j\boldsymbol{\xi}_i$, where $\boldsymbol{\alpha} = \alpha_i\boldsymbol{\xi}^i$ is an element of $T^*_{\boldsymbol{x}}M$. Especially the hamiltonian vector field (8.9) can be expressed as

$$\boldsymbol{h}_H = i_{\boldsymbol{d}H}\mathtt{J} = \mathtt{J}\cdot\boldsymbol{d}H \tag{10.2}$$

and the Poisson bracket as

$$\{F, G\}_{PB} = i_{\boldsymbol{d}F\boldsymbol{d}G}\mathtt{J} = (\boldsymbol{d}G\boldsymbol{d}F)\cdot\mathtt{J}, \tag{10.3}$$

so that the hamiltonian vector field $\boldsymbol{h}_H$ can be defined for all scalar functions F as

$$\boldsymbol{h}_H \cdot \boldsymbol{d}F = \{F, H\}_{PB}. \tag{10.4}$$

Equating (8.11) and (10.3) shows how Ω and $\mathtt{J}$ determine each other:

$$(\boldsymbol{h}_G\boldsymbol{h}_F)\cdot\Omega = (\boldsymbol{d}G\boldsymbol{d}F)\cdot\mathtt{J}. \tag{10.5}$$

Since a Poisson manifold can be odd-dimensional, the hamiltonian vector fields do not span in general the tangent space of the Poisson manifold. This suggests to define the range $\mathrm{ran}(\mathtt{J}(\boldsymbol{x}))$ of $\mathtt{J}(\boldsymbol{x})$ as the span of all tangent vectors that can be expressed as $\boldsymbol{\alpha}^\natural$ for a one-form $\boldsymbol{\alpha} \in T^*_{\boldsymbol{x}}M$. The range of $\mathtt{J}(\boldsymbol{x})$ is also the span of all hamiltonian vector fields at $\boldsymbol{x}$. The dimension of $\mathrm{ran}(\mathtt{J}(\boldsymbol{x}))$ is the rank of the Poisson manifold in $\boldsymbol{x}$ and equal to the rank of the matrix J^{ij}, which is an even number because of the anti-symmetry of J^{ij}. The even-dimensional vector space $\mathrm{ran}(\mathtt{J}(\boldsymbol{x}))$ is then the tangent space of a symplectic leaf in the point $\boldsymbol{x}$. The Poisson manifold is foliated by these symplectic leafs. Only when the rank of a Poisson manifold M is everywhere equal to $\dim(M)$ the Poisson manifold itself is a symplectic manifold.

The formalism developed so far can now directly be generalized to multivectors, which leads to Poisson calculus (see [29] and the references therein). The r-vector that corresponds to an r-form is given by

$$\left(A^{(r)}\right)^\natural = \frac{1}{r!} J^{k_1 i_1} \dots J^{k_r i_i} A_{i_1 \dots i_r} \boldsymbol{\xi}_{k_1} \dots \boldsymbol{\xi}_{k_r} \tag{10.6}$$

and in analogy to (4.53) one has $i_{A^{(r)}} B_{(s)} = \overline{A^{(r)}} \cdot B_{(s)}$, so that

$$\overline{\boldsymbol{\alpha}_1 \dots \boldsymbol{\alpha}_r} \cdot \left(A^{(r)}\right)^\natural = (-1)^r \overline{\boldsymbol{\alpha}_1^\natural \dots \boldsymbol{\alpha}_r^\natural} \cdot A^{(r)}. \tag{10.7}$$

It is then also possible to define a Poisson bracket for one-forms by

$$\{\boldsymbol{\alpha}, \boldsymbol{\beta}\}_{PB} = \mathscr{L}_{\boldsymbol{\alpha}^\natural} \boldsymbol{\beta} - \mathscr{L}_{\boldsymbol{\beta}^\natural} \boldsymbol{\alpha} + \boldsymbol{d}\big((\boldsymbol{\beta}\boldsymbol{\alpha}) \cdot \mathtt{J}\big), \tag{10.8}$$

so that $\{\boldsymbol{\alpha}, \boldsymbol{\beta}\}_{PB}^\natural = \left[\boldsymbol{\alpha}^\natural, \boldsymbol{\beta}^\natural\right]_{JLB}$. With this Poisson bracket one can further define an exterior differential $\tilde{\boldsymbol{d}}$ in analogy to (4.54) as

$$\begin{aligned} (\overline{\boldsymbol{\alpha}_1 \boldsymbol{\alpha}_2 \dots \boldsymbol{\alpha}_{r+1}}) \cdot \tilde{\boldsymbol{d}} A_{(r)} \quad = \quad & \sum_{n=1}^{r+1} (-1)^{n+1} (\boldsymbol{\alpha}_n^\natural \cdot \boldsymbol{\partial}) \left(\overline{\boldsymbol{\alpha}_1 \dots \check{\boldsymbol{\alpha}}_n \dots \boldsymbol{\alpha}_{r+1}}\right) \cdot A_{(r)} \\ & + \sum_{m<n} (-1)^{m+n} \left(\overline{\{\boldsymbol{\alpha}_m, \boldsymbol{\alpha}_n\}_{PB} \boldsymbol{\alpha}_1 \dots \check{\boldsymbol{\alpha}}_m \dots \check{\boldsymbol{\alpha}}_n \dots \boldsymbol{\alpha}_{r+1}}\right) \cdot A_{(r)}, \end{aligned} \tag{10.9}$$

which can also be written as $\tilde{\boldsymbol{d}} A_{(r)} = \left[\mathtt{J}, A_{(r)}\right]_{SNB}$.

The easiest non-constant Poisson tensor fulfilling (10.1) is a linear tensor

$$J^{ij}(\boldsymbol{x}) = C_k^{ij} x^k, \tag{10.10}$$

where the antisymmetry of J^{ij} and (10.1) ensure that the C_k^{ij} are structure constants of a Lie algebra. The corresponding Poisson bracket is the so called Lie-Poisson bracket

$$\{F, G\}_{LPB} = C_k^{ij} x^k \partial_i F \partial_j G. \tag{10.11}$$

The most fundamental example is the Lie-Poisson structure on $\mathfrak{g}^*$. For this purpose one considers the bivector space spanned by the basis bivectors $\mathtt{B}_i$ with bivector algebra (6.2) and its reciprocal basis with two-forms Θ^i, i.e. $\overline{\mathtt{B}_i} \cdot \Theta^j = \delta_i^j$. For scalar-valued functions F and G of general two-forms $\Theta = \theta_i \Theta^i$ a Lie-Poisson bracket is given by

$$\{F, G\}_{LPB}(\Theta) = C_{ij}^k \theta_k \frac{\partial F}{\partial \theta_i} \frac{\partial G}{\partial \theta_j} = \overline{(\mathtt{d}F \times \mathtt{d}G)} \cdot \Theta, \tag{10.12}$$

where $\mathtt{d}$ is the exterior differential in the bivector basis: $\mathtt{d} = \mathtt{B}_i \frac{\partial}{\partial \theta_i}$. In the $SO(3)$-case, where $\Theta^i = \mathtt{B}_i$ the Lie-Poisson bracket can be written as

$$\{F, G\}_{LPB}(\mathtt{B}) = \overline{\mathtt{B}} \cdot \big((I_{(3)} *_C \boldsymbol{d})F \times (I_{(3)} *_C \boldsymbol{d})G\big) = \overline{\mathtt{B}} \cdot (\mathtt{d}F \times \mathtt{d}G)\,. \tag{10.13}$$

The symplectic leafs induced by the symplectic foliation with the Lie-Poisson bracket on $\mathfrak{g}^*$ are the orbits of the coadjoint action of the corresponding group G on $\mathfrak{g}^*$. This can be seen if one considers a scalar linear function $H(\Theta) = \overline{\mathtt{B}} \cdot \Theta = b^i \theta_i$ on $\mathfrak{g}^*$ with $\mathtt{d}H = \mathtt{B}$. For the Lie-Poisson bracket one has then for any scalar function F on $\mathfrak{g}^*$:

$$\{F, H\}_{LPB}(\Theta) = \overline{(\mathtt{d}F \times \mathtt{d}H)} \cdot \Theta = -\overline{(\mathtt{B} \times \mathtt{d}F)} \cdot \Theta = -\overline{(\mathrm{ad}_{\mathtt{B}} \mathtt{d}F)} \cdot \Theta = -\overline{\mathtt{d}F} \cdot \mathrm{ad}^*_{\mathtt{B}} \Theta. \tag{10.14}$$

On the other hand one can define in analogy to (10.4) the hamiltonian bivector field $\mathtt{h}_H$ of the Hamilton function $H(\Theta)$ as

$$\overline{\mathtt{h}_H(\Theta)} \cdot \mathtt{d}F = \{F, H\}_{LPB}(\Theta) = \overline{(\mathtt{d}F \times \mathtt{d}H)} \cdot \Theta = -\overline{\mathrm{ad}^*_{\mathtt{B}} \Theta} \cdot \mathtt{d}F, \tag{10.15}$$

so that $\mathtt{h}_H(\Theta) = -\mathrm{ad}^*_{\mathtt{d}H} \Theta$. This means that the hamiltonian bivector fields $\mathtt{h}_H$ that span the tangent space of the symplectic leaf are, up to a sign, the generators of the coadjoint action determined by $\mathtt{B}$. If Θ varies now over the coadjoint orbit one can define a skew-symmetric bilinear form on the orbit by

$$\Omega_\Theta(\mathrm{ad}^*_{\mathtt{A}} \Theta, \mathrm{ad}^*_{\mathtt{B}} \Theta) = \overline{\mathtt{A} \times \mathtt{B}} \cdot \Theta, \tag{10.16}$$

which defines on the coadjoint orbit a symplectic structure, that is the restriction of the Lie-Poisson bracket to the orbit [26]. Ω_Θ can be seen as a generalized antisymmetric tensor of the form (4.76) that maps two bivectors into a scalar.

Of special interest is the hamiltonian action of a rotor on a Poisson vector manifold. The aim is to find the Hamilton function $P_{\mathtt{B}}$ of the vector field $\mathtt{B} \cdot \boldsymbol{x}$, that is induced according to (6.15) by the rotor left-action with bivector $\mathtt{B}$, i.e.

$$\boldsymbol{h}_{P_{\mathtt{B}}} = \mathtt{B} \cdot \boldsymbol{x}. \tag{10.17}$$

Since $\boldsymbol{h}_{P_{\mathtt{B}}} \cdot \boldsymbol{d}H = \{H, P_{\mathtt{B}}\}_{PB}$, it is possible to write the defining relation for $P_{\mathtt{B}}$ as

$$\{H, P_{\mathtt{B}}\}_{PB} = (\mathtt{B} \cdot \boldsymbol{x}) \cdot \boldsymbol{d}H, \tag{10.18}$$

for all scalar functions H. Furthermore one has for two bivectors $\mathtt{A}$ and $\mathtt{B}$ with (8.12) and (6.16)

$$\boldsymbol{h}_{\{P_{\mathtt{A}}, P_{\mathtt{B}}\}_{PB}} = \boldsymbol{h}_{P_{\mathtt{A} \times \mathtt{B}}}. \tag{10.19}$$

While in the symplectic case a symplectic vector field is always locally hamiltonian, in the Poisson case an infinitesimal Poisson automorphism is in general not locally hamiltonian. This means that if the rotor left-action is canonical, i.e. $\mathscr{L}_{\mathtt{B} \cdot \boldsymbol{x}} \mathtt{J} = 0$, there does not exist in general a function $P_{\mathtt{B}}$, that fulfills (10.17). The additional condition that $\mathtt{B} \cdot \boldsymbol{x}$ is also hamiltonian can be expressed with the momentum map. A momentum map is here a two-form $\Pi(\boldsymbol{x})$ with

$$i_{\mathtt{B}} \Pi = \overline{\mathtt{B}} \cdot \Pi = P_{\mathtt{B}}. \tag{10.20}$$

So if the hamiltonian vector field $\boldsymbol{h}_{P_{\mathtt{B}}}$ corresponding to the function $P_{\mathtt{B}} = \overline{\mathtt{B}} \cdot \Pi$ is the same as the vector field $\mathtt{B} \cdot \boldsymbol{x}$ induced by the rotor left-action, i.e. if one has $\boldsymbol{h}_{\overline{\mathtt{B}} \cdot \Pi} = (\mathtt{J} \cdot \boldsymbol{d}) \cdot (\overline{\mathtt{B}} \cdot \Pi) = \mathtt{B} \cdot \boldsymbol{x}$, then Π is a momentum map. If a momentum map of a rotor action exists and H is a Hamilton function that is invariant under the rotor action, then equation (10.18) reduces to $\{H, P_{\mathtt{B}}\}_{PB} = 0$ and the momentum map is a constant of the motion

described by H. This follows because $\{H, P_{\mathtt{B}}\}_{PB} = 0$ means that $P_{\mathtt{B}}$ is constant along the hamiltonian flow of H, which must then also be true for the left hand side of (10.20), i.e. for Π, because $\mathtt{B}$ is constant. This is the Noether theorem in the Poisson case.

If a hamiltonian action of a rotor group on a Poisson vector manifold is given, there are scalar functions $P_{\mathtt{B}_i}$ on the Poisson manifold with $\{P_i, P_j\}_{PB} = -C_{ij}^k P_k$ and $\left[\boldsymbol{h}_{P_i}, \boldsymbol{h}_{P_j}\right]_{JLB} = C_{ij}^k \boldsymbol{h}_{P_k}$, that generate the hamiltonian action. The momentum map is then

$$\Pi(\boldsymbol{x}) = P_{\mathtt{B}_i}(\boldsymbol{x})\Theta^i. \tag{10.21}$$

A momentum map $\Pi(\boldsymbol{x})$ that is determined by a hamiltonian group action is equivariant, i.e. it respects the rotor left-action on the vector manifold:

$$\Pi(R *_C \boldsymbol{x} *_C \overline{R}) = R *_C \Pi(\boldsymbol{x}) *_C \overline{R}, \tag{10.22}$$

which can also be written as

$$\overline{\mathrm{Ad}_R \mathtt{B}} \cdot \Pi(R *_C \boldsymbol{x} *_C \overline{R}) \equiv P_{\mathrm{Ad}_R \mathtt{B}}(R *_C \boldsymbol{x} *_C \overline{R}) = P_{\mathtt{B}}(\boldsymbol{x}) \equiv \overline{\mathtt{B}} \cdot \Pi(\boldsymbol{x}), \tag{10.23}$$

To see that the momentum map (10.21) is equivariant, it suffices to show the infinitesimal version of (10.22), namely $\left(\boldsymbol{h}_{P_{\mathtt{B}_j}} \cdot \boldsymbol{d}\right) P_{\mathtt{B}_i}\Theta^i = \mathtt{B}_j \times \Pi$, which immediately reduces to $-C_{ij}^k P_{\mathtt{B}_k}\Theta^i = P_{\mathtt{B}_i} C_{jk}^i \Theta^k$.

Infinitesimal equivariance [26] implies that $P_{\mathtt{A}\times\mathtt{B}} = \{P_{\mathtt{A}}, P_{\mathtt{B}}\}_{PB}$. Then it is easy to see that equivariant momentum maps are Poisson maps, i.e. for scalar-valued functions F and G on $\mathfrak{g}^*$ one has

$$\{F, G\}_{LPB}(\Pi(\boldsymbol{x})) = \{F(\Pi(\boldsymbol{x})), G(\Pi(\boldsymbol{x}))\}_{PB}. \tag{10.24}$$

To prove this one shows that the left hand side of (10.24) can be written as

$$\{F, G\}_{LPB}(\Pi(\boldsymbol{x})) = \overline{\mathtt{d}F \times \mathtt{d}G} \cdot \Pi(\boldsymbol{x}) = P_{\mathtt{d}F\times\mathtt{d}G} = \{P_{\mathtt{d}G}, P_{\mathtt{d}H}\}_{PB}, \tag{10.25}$$

where one uses in the last step infinitesimal equivariance. The right hand side of (10.24) gives the same result:

$$\{F(\Pi(\boldsymbol{x})), G(\Pi(\boldsymbol{x}))\}_{PB} = J^{ij}\partial_i F(\Pi(\boldsymbol{x}))\partial_j G(\Pi(\boldsymbol{x})) = J^{ij}\partial_i P_{\mathtt{d}F}\partial_j P_{\mathtt{d}G} = \{P_{\mathtt{d}F}, P_{\mathtt{d}G}\}_{PB}, \tag{10.26}$$

with $\partial_i F(\Pi(\boldsymbol{x})) = \overline{\mathtt{d}F} \cdot \partial_i \Pi(\boldsymbol{x}) = \partial_i \left(\overline{\mathtt{d}F} \cdot \Pi(\boldsymbol{x})\right) = \partial_i P_{\mathtt{d}F}$.

A special case for a momentum map is the momentum map of the cotangent lift of a rotor action on a vector manifold $\boldsymbol{q} = q^a(q^i)\boldsymbol{\sigma}_a$. In order to find this momentum map one first states that it is possible to find for a tangent vector field $\boldsymbol{a}(\boldsymbol{q}) = a^i \xi_i^a \boldsymbol{\sigma}_a$ a function $P_{\boldsymbol{a}}(q^i, p_i) = P_{\boldsymbol{a}}(\boldsymbol{q} + \boldsymbol{\pi})$ on the cotangent bundle, which is given with the projection operator (8.16) as:

$$P_{\boldsymbol{a}}(q^i, p_i) = T\pi_{\boldsymbol{q}}(\boldsymbol{a}) \cdot (\boldsymbol{q} + \boldsymbol{\pi}) = a^j \xi_j^a \boldsymbol{\tau}_a \cdot (q^b \boldsymbol{\sigma}_b + p_k \xi_b^k \boldsymbol{\tau}^b) = a^j(q^i)p_j. \tag{10.27}$$

These functions form an algebra on the cotangent bundle, i.e. $\{P_{\boldsymbol{a}}, P_{\boldsymbol{b}}\}_{PB} = -P_{[\boldsymbol{a},\boldsymbol{b}]_{JLB}}$. The rotor action of a rotor $R(t) = e_{*_C}^{\frac{t}{2}\mathtt{B}}$ on the vector manifold $\boldsymbol{q}$ induces a flow $\boldsymbol{q}(t) = R(t) *_C \boldsymbol{q} *_C \overline{R(t)}$ and a tangential vector field $\boldsymbol{b} = \mathtt{B} \cdot \boldsymbol{q}$. The inverse cotangent lift of this rotor action is

$$(\boldsymbol{q} + \boldsymbol{\pi})(t) = \overline{R_{\text{lifted}}(-t)} *_C (\boldsymbol{q} + \boldsymbol{\pi}) *_C R_{\text{lifted}}(-t) = R_{\text{lifted}}(t) *_C (\boldsymbol{q} + \boldsymbol{\pi}) *_C \overline{R_{\text{lifted}}(t)}, \tag{10.28}$$

which induces on the cotangent bundle a tangent vector field $\boldsymbol{b}_{\text{lifted}} = \mathtt{B}_{\text{lifted}} \cdot (\boldsymbol{q}+\boldsymbol{\pi})$. The vector field $\boldsymbol{b}_{\text{lifted}}$ is then the hamiltonian vector field of $P_{\boldsymbol{b}}$, i.e. $\boldsymbol{b}_{\text{lifted}} = \boldsymbol{h}_{P_{\boldsymbol{b}}}$. This can be proved very easily if one considers that the cotangent lift of a rotor action leaves the canonical one-form invariant, i.e. $\mathscr{L}_{\boldsymbol{b}_{\text{lifted}}}\boldsymbol{\theta} = 0$. Cartan's magic formula (4.55) gives then

$$\boldsymbol{b}_{\text{lifted}} \cdot \Omega = -i_{\boldsymbol{b}_{\text{lifted}}} \boldsymbol{d}\boldsymbol{\theta} = \boldsymbol{d} i_{\boldsymbol{b}_{\text{lifted}}} \boldsymbol{\theta} = \boldsymbol{d}(\boldsymbol{b}_{\text{lifted}} \cdot \boldsymbol{\theta}). \tag{10.29}$$

On the other hand one has with (8.14) and (8.16)

$$\boldsymbol{b}_{\text{lifted}} \cdot \boldsymbol{\theta}(\boldsymbol{q}+\boldsymbol{\pi}) = T\pi_{\boldsymbol{q}}(\boldsymbol{b}_{\text{lifted}}) \cdot \boldsymbol{\pi} = T\pi_{\boldsymbol{q}}(\boldsymbol{b}) \cdot \boldsymbol{\pi} = P_{\boldsymbol{b}}(\boldsymbol{q}+\boldsymbol{\pi}). \tag{10.30}$$

Putting this into (10.29) gives

$$\boldsymbol{b}_{\text{lifted}} \cdot \Omega = \boldsymbol{d} P_{\boldsymbol{b}}, \tag{10.31}$$

which shows that $\boldsymbol{b}_{\text{lifted}}$ is the hamiltonian vector field of $P_{\boldsymbol{b}}$.

The momentum map of the cotangent lift of a rotor action on the vector manifold $\boldsymbol{q}$ is then given for $\boldsymbol{b} = \mathtt{B} \cdot \boldsymbol{q}$ by

$$\overline{\mathtt{B}} \cdot \Pi(\boldsymbol{q}+\boldsymbol{\pi}) = T\pi_{\boldsymbol{q}}(\mathtt{B} \cdot \boldsymbol{q}) \cdot (\boldsymbol{q}+\boldsymbol{\pi}) = P_{\boldsymbol{b}}(\boldsymbol{q}+\boldsymbol{\pi}). \tag{10.32}$$

Moreover this momentum map is also equivariant:

$$\begin{aligned}
\overline{\mathtt{B}} \cdot \Pi(R_{\text{lifted}} *_C (\boldsymbol{q}+\boldsymbol{\pi}) *_C \overline{R_{\text{lifted}}}) &= T\pi_{\boldsymbol{q}}\left(\mathtt{B} \cdot (R *_C \boldsymbol{q} *_C \overline{R})\right) \cdot (R_{\text{lifted}} *_C (\boldsymbol{q}+\boldsymbol{\pi}) *_C \overline{R_{\text{lifted}}}) && (10.33)\\
&= T\pi_{\boldsymbol{q}}(\mathrm{Ad}_{\overline{R}}\mathtt{B} \cdot \boldsymbol{q}) \cdot (\boldsymbol{q}+\boldsymbol{\pi}) && (10.34)\\
&= \overline{\mathrm{Ad}_{\overline{R}}\mathtt{B}} \cdot \Pi(\boldsymbol{q}+\boldsymbol{\pi}). && (10.35)
\end{aligned}$$

A simple example is the action of the rotation group on a three dimensional euclidian vector space with vectors $\boldsymbol{q} = q^i\boldsymbol{\eta}_i$ for $i = 1, 2, 3$. The tangent bundle is then a six dimensional euclidian vector space with vectors $\boldsymbol{q}+\boldsymbol{\pi} = q^i\boldsymbol{\eta}_i + p_i\boldsymbol{\rho}^i$ and a canonical symplectic structure $\Omega = \boldsymbol{\eta}^i\boldsymbol{\rho}_i$. A rotation on the $\boldsymbol{q}$-space is generated by the bivectors $\mathtt{B}_i = \frac{1}{2}\varepsilon_{ijk}\boldsymbol{\eta}_j\boldsymbol{\eta}_k$. For example a rotation around the $\boldsymbol{\eta}_3$-axis is generated by $\mathtt{B}_3 = \boldsymbol{\eta}_1\boldsymbol{\eta}_2$ and the corresponding vector field is $\boldsymbol{b}_3 = \mathtt{B}_3 \cdot \boldsymbol{q} = q^2\boldsymbol{\eta}_1 - q^1\boldsymbol{\eta}_2$. The lifted rotation is a rotation that acts in the $\boldsymbol{\rho}_i$-space just the same way as in the $\boldsymbol{\eta}_i$-space, the lifted generator is then $\mathtt{B}_3^{\text{lifted}} = \boldsymbol{\eta}_1\boldsymbol{\eta}_2 + \boldsymbol{\rho}^1\boldsymbol{\rho}^2$ and the corresponding lifted vector field is given by

$$\boldsymbol{b}_3^{\text{lifted}} = \mathtt{B}_3^{\text{lifted}} \cdot (\boldsymbol{q}+\boldsymbol{\pi}) = q^2\boldsymbol{\eta}_1 - q^1\boldsymbol{\eta}_2 + p_2\boldsymbol{\rho}^1 - p_1\boldsymbol{\rho}^2. \tag{10.36}$$

The Hamilton function $P_{\mathtt{B}_3}$ that generates this vector field fulfills $\boldsymbol{b}_3^{\text{lifted}} \cdot \Omega = \boldsymbol{d}P_{\mathtt{B}_3}$ or

$$p^2\boldsymbol{\eta}^1 - p^1\boldsymbol{\eta}^2 - q^2\boldsymbol{\rho}^1 + q^1\boldsymbol{\rho}^2 = \boldsymbol{\eta}^1\frac{\partial P_{\mathtt{B}_3}}{\partial q^1} + \boldsymbol{\eta}^2\frac{\partial P_{\mathtt{B}_3}}{\partial q^2} + \boldsymbol{\rho}_1\frac{\partial P_{\mathtt{B}_3}}{\partial p_1} + \boldsymbol{\rho}_2\frac{\partial P_{\mathtt{B}_3}}{\partial p_2}, \tag{10.37}$$

which is solved by the angular momentum function. The angular momentum functions $P_{\mathtt{B}_i} = \varepsilon_{ij}^k q^j p_k$ are the generators of the active rotations, that rotate the q^i just as the p_i coefficients. They form the algebra $\{P_{\mathtt{B}_i}, P_{\mathtt{B}_j}\}_{PB} = \varepsilon_{ijk}P_{\mathtt{B}_k}$, so that there is a hamiltonian action of the rotations on the six dimensional symplectic space. The momentum map $\Pi(q^i, p_i) = P_{\mathtt{B}_j}(q^i, p_i)\Theta^j$ is just the angular momentum bivector $\mathtt{L} = \boldsymbol{q}\boldsymbol{p}$ and connects the generators of the active and passive rotations.

Another simple example is the circle action of S^1 on S^2 [25]. The two-dimensional sphere $\boldsymbol{x}(\theta, \varphi) = \sin\theta\cos\varphi\boldsymbol{\sigma}_1 + \sin\theta\sin\varphi\boldsymbol{\sigma}_2 + \cos\theta\boldsymbol{\sigma}_3$ is a symplectic vector manifold with the symplectic two-form

$$\Omega = x^1\boldsymbol{\sigma}^2\boldsymbol{\sigma}^3 + x^2\boldsymbol{\sigma}^3\boldsymbol{\sigma}^1 + x^3\boldsymbol{\sigma}^1\boldsymbol{\sigma}^2\Big|_{S^2} = \sin\theta\boldsymbol{\xi}^\theta\boldsymbol{\xi}^\varphi, \tag{10.38}$$

which is the volume form on the S^2. A left rotation around the $\boldsymbol{\sigma}_3$-axis is generated by $\mathtt{B} = -\boldsymbol{\sigma}_1\boldsymbol{\sigma}_2$ and induces on S^2 the vector field

$$\mathtt{B} \cdot \boldsymbol{x} = \sin\theta\cos\varphi\boldsymbol{\sigma}_2 - \sin\theta\sin\varphi\boldsymbol{\sigma}_1 = \partial_\varphi \boldsymbol{x} = \boldsymbol{\xi}_\varphi. \tag{10.39}$$

The Hamilton function $P_{\mathtt{B}}$ that generates this vector field fulfills according to (8.8) the equation $\boldsymbol{\xi}_\varphi \cdot \Omega = \boldsymbol{d}P_{\mathtt{B}}$, or

$$-\sin\theta\boldsymbol{\xi}^\theta = \boldsymbol{\xi}^\varphi\partial_\varphi P_{\mathtt{B}} + \boldsymbol{\xi}^\theta\partial_\theta P_{\mathtt{B}}, \tag{10.40}$$

which is solved by $P_{\mathtt{B}} = \cos\theta = x^3$.

Applying now the concepts discussed so far to the cotangent space of a group manifold T^*G, which is a vector manifold with vectors $\boldsymbol{r} + \boldsymbol{\vartheta}$, one arrives at the Lie-Poisson reduction [26]. As seen above the rotors act on the group vector manifold with a left translation ℓ_R which induces the tangential maps $T\ell_R$ and $T^*\ell_R$. A scalar function $F(\boldsymbol{r}+\boldsymbol{\vartheta}) = F(R,\dot{R})$ on T^*G is left invariant if $F \circ T^*\ell_R = F$. Such left invariant functions can be identified with reduced functions on $\mathfrak{g}$, i.e. $F(\boldsymbol{r}+\boldsymbol{\vartheta}) = F(R,\dot{R}) = F(1, \overline{R} *_C \dot{R}) = f(\Theta)$, where $\overline{R} *_C \dot{R}$ is an element of the bivector algebra that can also be expressed in the dual basis. This reduction can now be described with the momentum map $\Pi : T^*G \to \mathfrak{g}^*$, i.e. $F(\boldsymbol{r}+\boldsymbol{\vartheta}) = f(\Pi(\boldsymbol{r}+\boldsymbol{\vartheta}))$. One has then a Poisson map between the Poisson bracket of left invariant functions on T^*G and the Lie-Poisson bracket of reduced functions on $\mathfrak{g}^*$. In this way a left invariant Hamilton function on T^*G induces a Lie-Poisson dynamic on $\mathfrak{g}^*$. This will be explained for the example of the rigid body in the next section.

11 The rigid Body

The rigid body is an example where the formalism described above can be shown to work very effectively. If one considers a free rigid body $\mathcal{B}$ in a three-dimensional ambient space spanned by the basis vectors $\boldsymbol{\sigma}_a$ and a body-fixed coordinate system $\boldsymbol{\xi}_i(t)$, a point of the body in the ambient space is given by

$$\boldsymbol{x}(t) = R(t) *_C \boldsymbol{x}_{\mathcal{B}} *_C \overline{R}(t), \tag{11.1}$$

where $\boldsymbol{x}_{\mathcal{B}}$ is the vector in the body-fixed system. The velocity is then given by

$$\begin{aligned}
\dot{\boldsymbol{x}} &= \dot{R} *_C \boldsymbol{x}_{\mathcal{B}} *_C \overline{R} + R *_C \boldsymbol{x}_{\mathcal{B}} *_C \dot{\overline{R}} && (11.2)\\
&= R *_C (\overline{R} *_C \dot{R} *_C \boldsymbol{x}_{\mathcal{B}} - \boldsymbol{x}_{\mathcal{B}} *_C \overline{R} *_C \dot{R}) *_C \overline{R} && (11.3)\\
&= \dot{R} *_C \overline{R} *_C \boldsymbol{x} - \boldsymbol{x} *_C \dot{R} *_C \overline{R} && (11.4)\\
&= 2(\dot{R} *_C \overline{R}) \cdot \boldsymbol{x}, && (11.5)
\end{aligned}$$

using $\overline{R} *_C R = 1 \Rightarrow \dot{\overline{R}} *_C R + \overline{R} *_C \dot{R} = 0$. And for the body-fixed velocity one obtains

$$\dot{\boldsymbol{x}}_{\mathcal{B}} = \overline{R} *_C \dot{\boldsymbol{x}} *_C R = 2(\overline{R} *_C \dot{R}) \cdot \boldsymbol{x}_{\mathcal{B}}. \tag{11.6}$$

On the other side one has $\dot{\boldsymbol{x}} = \boldsymbol{\omega} \times \boldsymbol{x}$, where $\boldsymbol{\omega}$ is the axial vector of angular velocity. Using that the vector cross product can be written as $\boldsymbol{a} \times \boldsymbol{b} = -(I_{(3)} *_C \boldsymbol{a}) \cdot \boldsymbol{b}$ this leads to

$$\dot{\boldsymbol{x}} = -(I_{(3)} *_C \boldsymbol{\omega}) \cdot \boldsymbol{x} = -\mathtt{W} \cdot \boldsymbol{x}, \tag{11.7}$$

where

$$\mathtt{W} = -2\dot{R} *_C \overline{R} = I_{(3)} *_C \boldsymbol{\omega} \tag{11.8}$$

is the angular velocity bivector that generates the rotation. Equation (11.8) can be rewritten to obtain the rotor equation

$$\dot{R} = -\frac{1}{2}\mathtt{W} *_C R, \tag{11.9}$$

which integrates for constant angular velocity to $R = e_{*_C}^{-\frac{t}{2}\mathtt{W}}$. With the angular velocity bivector (11.8) one can also write (11.7) as

$$\dot{\boldsymbol{x}} = (R *_C \boldsymbol{x}_{\mathcal{B}} *_C \overline{R}) \cdot \mathtt{W} = R *_C (\boldsymbol{x}_{\mathcal{B}} \cdot \mathtt{W}_{\mathcal{B}}) *_C \overline{R}, \tag{11.10}$$

where $\mathtt{W}_{\mathcal{B}} = \overline{R} *_C \mathtt{W} *_C R = -2\overline{R} *_C \dot{R}$, so that the rotor equation becomes $\dot{R} = -\frac{1}{2} R *_C \mathtt{W}_{\mathcal{B}}$.

The angular momentum bivector is given by

$$\begin{aligned} \mathtt{L} &= \int d^3x\, \rho(\boldsymbol{x})\, \boldsymbol{x}\dot{\boldsymbol{x}} = \int d^3x_{\mathcal{B}}\, \rho(\boldsymbol{x}_{\mathcal{B}})\, (R *_C \boldsymbol{x}_{\mathcal{B}} *_C \overline{R})(R *_C (\boldsymbol{x}_{\mathcal{B}} \cdot \mathtt{W}_{\mathcal{B}}) *_C \overline{R}) && (11.11) \\ &= R *_C \left(\int d^3x_{\mathcal{B}}\, \rho(\boldsymbol{x}_{\mathcal{B}})\, \boldsymbol{x}_{\mathcal{B}}(\boldsymbol{x}_{\mathcal{B}} \cdot \mathtt{W}_{\mathcal{B}}) \right) *_C \overline{R} = R *_C \mathtt{I}(\mathtt{W}_{\mathcal{B}}) *_C \overline{R}, && (11.12) \end{aligned}$$

where the bivector-valued function of a bivector

$$\mathtt{I}(\mathtt{B}) = \int d^3x_{\mathcal{B}}\, \rho(\boldsymbol{x}_{\mathcal{B}})\, \boldsymbol{x}_{\mathcal{B}}(\boldsymbol{x}_{\mathcal{B}} \cdot \mathtt{B}), \tag{11.13}$$

corresponds to the inertial tensor. The equation of motion of the free rigid body can be obtained from

$$\begin{aligned} 0 = \dot{\mathtt{L}} &= \dot{R} *_C \mathtt{I}(\mathtt{W}_{\mathcal{B}}) *_C \overline{R} + R *_C \mathtt{I}(\mathtt{W}_{\mathcal{B}}) *_C \dot{\overline{R}} + R *_C \mathtt{I}(\dot{\mathtt{W}}_{\mathcal{B}}) *_C \overline{R} && (11.14) \\ &= R *_C \left(\mathtt{I}(\dot{\mathtt{W}}_{\mathcal{B}}) - \mathtt{W}_{\mathcal{B}} \times \mathtt{I}(\mathtt{W}_{\mathcal{B}})\right) *_C \overline{R} && (11.15) \end{aligned}$$

as $\mathtt{I}(\dot{\mathtt{W}}_{\mathcal{B}}) - \mathtt{W}_{\mathcal{B}} \times \mathtt{I}(\mathtt{W}_{\mathcal{B}}) = 0$, which are the Euler equations.

The other possibility to derive the equations of motion is to use the Lagrange or Hamilton formalism. The kinetic energy of the free rigid body can be written with (11.6) as

$$\begin{aligned} T &= \frac{1}{2} \int d^3x_{\mathcal{B}}\, \rho(\boldsymbol{x}_{\mathcal{B}})\, |2(\overline{R} *_C \dot{R}) \cdot \boldsymbol{x}_{\mathcal{B}}|^2 && (11.16) \\ &= \frac{1}{2} \int d^3x_{\mathcal{B}}\, \rho(\boldsymbol{x}_{\mathcal{B}})\, |\mathtt{W}_{\mathcal{B}} \cdot \boldsymbol{x}_{\mathcal{B}}|^2 && (11.17) \\ &= \frac{1}{2}\overline{\mathtt{W}_{\mathcal{B}}} \cdot \mathtt{I}(\mathtt{W}_{\mathcal{B}}) && (11.18) \\ &= \frac{1}{2}\overline{\mathtt{W}} \cdot \mathtt{L}. && (11.19) \end{aligned}$$

Equation (11.16) is the left invariant Lagrangian $L(R, \dot{R})$ and (11.18) the reduced Lagrangian $l(\mathtt{W}_{\mathcal{B}})$ of the free rigid body. This means that the dynamics is transferred by (11.1) from the vectors $\boldsymbol{x}(t)$ to the rotors or the generating bivectors, i.e. one considers the dynamics on the rotor group or the bivector algebra respectively, which is the same idea that underlies the Kustaanheimo-Stiefel transformation.

The question is now how to vary the corresponding Lagrangians. In analogy to the matrix representation [26] one has

$$\begin{aligned} \delta\mathtt{W}_{\mathcal{B}} = \delta(-2\overline{R} *_C \dot{R}) &= 2\overline{R} *_C \delta R *_C \overline{R} *_C \dot{R} - 2\overline{R} *_C \delta\dot{R} && (11.20) \\ &= -\overline{R} *_C \delta R *_C \mathtt{W}_{\mathcal{B}} - 2\overline{R} *_C \delta\dot{R} && (11.21) \end{aligned}$$

and defining the bivector $\mathtt{B} = 2\overline{R} *_C \delta R$ so that

$$\dot{\mathtt{B}} = \mathtt{W}_{\mathcal{B}} *_C \frac{1}{2}\mathtt{B} + 2\overline{R} *_C \delta\dot{R}, \tag{11.22}$$

one obtains

$$\delta\mathtt{W}_{\mathcal{B}} = -\dot{\mathtt{B}} + \mathtt{W}_{\mathcal{B}} \times \mathtt{B}. \tag{11.23}$$

The variation

$$\begin{aligned} 0 = \delta l(\mathtt{W}_{\mathcal{B}}) = \delta \int dt\, \frac{1}{2}\overline{\mathtt{W}_{\mathcal{B}}} \cdot \mathtt{I}(\mathtt{W}_{\mathcal{B}}) &= \int dt \int d^3x_{\mathcal{B}}\, \rho(\boldsymbol{x}_{\mathcal{B}})\, \overline{\delta\mathtt{W}_{\mathcal{B}}} \cdot [\boldsymbol{x}_{\mathcal{B}}(\boldsymbol{x}_{\mathcal{B}} \cdot \mathtt{W}_{\mathcal{B}})] && (11.24)\\ &= \int dt\, \overline{\mathtt{I}(\mathtt{W}_{\mathcal{B}})} \cdot \big(-\dot{\mathtt{B}} + \mathtt{W}_{\mathcal{B}} \times \mathtt{B}\big) && (11.25)\\ &= \int dt\, \big[\mathtt{I}(\dot{\mathtt{W}}_{\mathcal{B}}) + \mathtt{I}(\mathtt{W}_{\mathcal{B}}) \times \mathtt{W}_{\mathcal{B}}\big] \cdot \overline{\mathtt{B}}, && (11.26) \end{aligned}$$

leads then again to the Euler equations, where one uses in (11.24)

$$\overline{\mathtt{W}_{\mathcal{B}}} \cdot [\boldsymbol{x}_{\mathcal{B}}(\boldsymbol{x}_{\mathcal{B}} \cdot \delta\mathtt{W}_{\mathcal{B}})] = \overline{\delta\mathtt{W}_{\mathcal{B}}} \cdot [\boldsymbol{x}_{\mathcal{B}}(\boldsymbol{x}_{\mathcal{B}} \cdot \mathtt{W}_{\mathcal{B}})] \tag{11.27}$$

and in (11.25) equation (11.23).

So given a left invariant rotor Lagrangian $L(R, \dot{R})$ and its reduction to the bivector algebra $l(\mathtt{W}_{\mathcal{B}})$, the variation of $L(R, \dot{R})$ corresponds to the variation of $l(\mathtt{W}_{\mathcal{B}})$ for variations $\delta\mathtt{W}_{\mathcal{B}} = -\dot{\mathtt{B}} + \mathtt{W}_{\mathcal{B}} \times \mathtt{B}$, where $\mathtt{B}$ is a bivector that vanishes at the endpoints. The Euler-Lagrange equation for the rotor corresponds to the bivector equation

$$\frac{d}{dt}\frac{\delta l}{\delta\mathtt{W}_{\mathcal{B}}} = \mathtt{W}_{\mathcal{B}} \times \frac{\delta l}{\delta\mathtt{W}_{\mathcal{B}}}. \tag{11.28}$$

The Euler-Poincaré reconstruction of the rotor from the bivector $\mathtt{W}_{\mathcal{B}}$ can then be done with the rotor equation and in a last step the dynamics $\boldsymbol{x}(t)$ is reobtained by (11.1).

In the Hamilton formalism the analogous construction is called Lie-Poisson reduction and can also be done in the rotor case. The Hamiltonian (11.18) of the free rigid body can be written as

$$H = \frac{1}{2}\left(\frac{L_{\mathcal{B}1}^2}{I_1} + \frac{L_{\mathcal{B}2}^2}{I_2} + \frac{L_{\mathcal{B}3}^2}{I_3}\right). \tag{11.29}$$

With the Lie-Poisson bracket (10.13)

$$\{F, G\}_{LPB}(\mathtt{L}_{\mathcal{B}}) = \overline{\mathtt{L}_{\mathcal{B}}} \cdot \big((I_{(3)} *_C \boldsymbol{\nabla} F) \times (I_{(3)} *_C \boldsymbol{\nabla} G)\big) = \overline{\mathtt{L}_{\mathcal{B}}} \cdot (\mathtt{d}F \times \mathtt{d}G) \tag{11.30}$$

the Euler equations are obtained by $\dot{L}_{\mathcal{B}i} = \{L_{\mathcal{B}i}, H\}_{LPB}$. They preserve the coadjoint orbit, i.e. the Casimir function $|\mathtt{L}_{\mathcal{B}}|^2$ is a constant of motion: $\{(L_{\mathcal{B}1}^2 + L_{\mathcal{B}2}^2 + L_{\mathcal{B}3}^2), H\}_{LPB} = 0$. The conserved quantity that results from the left invariance is the angular momentum, which follows from the calculation in (11.14) and (11.15).

The procedure described above is the bivector version of the Poincaré equation [30]. In order to derive the Poincaré equation one considers a vector manifold $\boldsymbol{x}(q^i)$ with coordinate basis vectors $\boldsymbol{\xi}_i = \partial_i \boldsymbol{x}$ and non-coordinate basis vectors $\boldsymbol{\vartheta}_r = \vartheta_r^i \boldsymbol{\xi}_i$. For a scalar-valued function $f(q^i(t))$ on a trajectory $\boldsymbol{q}(t) = \boldsymbol{x}(q^i(t))$ one has $\frac{d}{dt} f = \dot{q}^i \partial_i f$. In the non-coordinate basis the coefficients are $s^r = \vartheta_i^r \dot{q}^i$, so that $\frac{d}{dt} = s^r \partial_r$. On the

other hand the variation of the trajectory $\boldsymbol{q}(t) = \boldsymbol{q}(t, u = 0)$ is given by $\delta q^i = \frac{d}{du}\big|_{u=0} q^i(t, u) = w^i$, where the coefficients in the non-coordinate basis are $w^r = \vartheta^r_i w^i$. From the condition that the operators

$$\frac{d}{dt} = \boldsymbol{s} \cdot \boldsymbol{\partial} = s^r \partial_r \qquad \text{and} \qquad \frac{d}{du} = \boldsymbol{w} \cdot \boldsymbol{\partial} = w^r \partial_r \tag{11.31}$$

commute it follows that

$$\frac{d}{du}\boldsymbol{s} = \frac{d}{dt}\boldsymbol{w} + [\boldsymbol{s}, \boldsymbol{w}]_{JLB} \,. \tag{11.32}$$

This equation can now be used for varying the Lagrange function $L(q^i(t, u), s^r(t, u))$:

$$\begin{aligned} 0 = \delta S &= \int_a^b dt \left[\frac{\partial L}{\partial q^i} \frac{\partial q^i}{\partial u} + \frac{\partial L}{\partial s^r} \left(\frac{d}{dt} w^r + C^r_{st} s^s w^t \right) \right]_{u=0} && (11.33) \\ &= \int_a^b dt \left[\left(\partial_r L + \frac{\partial L}{\partial s^s} s^t C^s_{tr} - \frac{d}{dt} \frac{\partial L}{\partial s^r} \right) w^r + \frac{d}{dt} \left(\frac{\partial L}{\partial s^r} w^r \right) \right]_{u=0} , && (11.34) \end{aligned}$$

from which the Poincaré equation follows:

$$\frac{d}{dt}\frac{\partial L}{\partial s^r} - \frac{\partial L}{\partial s^s} s^t C^s_{tr} = \partial_r L. \tag{11.35}$$

If the configuration space is a rotor group the Lagrange function is $L = L(R, \dot{R})$ and one has to vary $R(t, u)$. Instead of vectors $\boldsymbol{s}$ and $\boldsymbol{w}$ the variations are described by bivectors

$$\mathtt{s} = 2\overline{R} *_C \dot{R} \qquad \text{and} \qquad \mathtt{w} = 2\overline{R} *_C \delta R, \tag{11.36}$$

so that the operators (11.31) are now expressed as $\frac{d}{dt} = \overline{\mathtt{s}} \cdot \mathtt{d}$ and $\frac{d}{du} = \overline{\mathtt{w}} \cdot \mathtt{d}$. It follows further that

$$\begin{aligned} \frac{d\mathtt{s}}{du} &= -2\overline{R} *_C \delta R *_C \overline{R} *_C \dot{R} + 2\overline{R} *_C \delta\dot{R} = -\frac{1}{2}\mathtt{w} *_C \mathtt{s} + 2\overline{R} *_C \delta\dot{R}, && (11.37) \\ \frac{d\mathtt{w}}{dt} &= -2\overline{R} *_C \dot{R} *_C \overline{R} *_C \delta R + 2\overline{R} *_C \delta\dot{R} = -\frac{1}{2}\mathtt{s} *_C \mathtt{w} + 2\overline{R} *_C \delta\dot{R}. && (11.38) \end{aligned}$$

Equating the expressions for $2\overline{R} *_C \delta\dot{R}$ gives the bivector analog of (11.32):

$$\frac{d}{du}\mathtt{s} = \frac{d}{dt}\mathtt{w} + \mathtt{s} \times \mathtt{w}, \tag{11.39}$$

so that the variation of the action

$$\begin{aligned} 0 = \delta S &= \int_a^b dt\, \delta L && (11.40) \\ &= \int_a^b dt \left[\overline{\mathtt{w}} \cdot \mathtt{d}L + \overline{\frac{\delta L}{\delta \mathtt{s}}} \cdot \left(\frac{d}{dt}\mathtt{w} + \mathtt{s} \times \mathtt{w} \right) \right]_{u=0} && (11.41) \\ &= \int_a^b dt \left[\overline{\mathtt{w}} \cdot \left(\mathtt{d}L - \frac{d}{dt}\frac{\delta L}{\delta \mathtt{s}} + \frac{\delta L}{\delta \mathtt{s}} \times \mathtt{s} \right) + \frac{d}{dt} \left(\overline{\mathtt{w}} \cdot \frac{\delta L}{\delta \mathtt{s}} \right) \right]_{u=0} , && (11.42) \end{aligned}$$

leads now to the bivector version of the Poincaré equation

$$\frac{d}{dt}\frac{\delta L}{\delta \mathtt{s}} - \frac{\delta L}{\delta \mathtt{s}} \times \mathtt{s} = \mathtt{d}L. \tag{11.43}$$

In the same way the Hamilton formalism is transferred from the vector to a bivector basis. The Hamilton equations $\dot{z}^i = \{z^i, H\}_{PB}$ in the bivector case, i.e. for a Hamilton function $H(\mathtt{z})$ with a bivector $\mathtt{z} = z^i \mathtt{B}_i$ are obtained by using the Lie-Poisson bracket instead of the Poisson bracket. In the $\mathfrak{so}(3)$-case the Hamilton equations read then

$$\dot{\mathtt{z}} = \mathtt{z} \times \mathtt{d}H = -\mathrm{ad}^*_{\mathtt{d}H}\mathtt{z}. \tag{11.44}$$

Conclusions

Comparing classical and quantum mechanics there are two formal breaks. The first one is that classical mechanics is formulated on the phase space, while quantum mechanics is formulated on a Hilbert space. This formal break is overcome by the bosonic star product formalism that describes quantum mechanics on the phase space. The second formal break is that classical mechanics is formulated conventionally in the Gibbs-Heavyside tuple vector formalism, while in quantum mechanics one is using actually a Clifford calculus in order to take care of the spin degrees of freedom. The Gibbs-Heavyside tuple formalism ignores the basis vectors and their naturally given Clifford structure. Unfortunately the basis vectors and their algebraic structure play an essential role if there is curvature or non-commutativity. And so the basis vectors had to be reintroduced in the formalism a posteriori which then naturally leads to a multivector formalism. The basis vectors appear for example as Dirac matrices, as differential forms or as Grassmann numbers. These different formalisms are notationally inconsistent and incomplete. For example exterior calculus is restricted to homogenous multivectors and in superanalysis there is no Clifford structure. In the case of Dirac matrices sticking to a tuple formalism has the disadvantage that one has to construct an unphysical spinor space in which the Clifford structure is represented by matrices. A complete Clifford multivector formalism was on the other hand developed physically in the context of Dirac theory by Hestenes and Kähler and in the context of phase space calculus by Gozzi and Reuter. The full multivector formalism can now be described with the star product formalism as deformed superanalysis and so a formal supersymmetry is introduced in the formalism. The combination of star products and geometric algebra leads to a formalism that unifies the different geometric calculi on commutative and noncommutative spaces, on flat and curved spaces, on tangent and cotangent spaces and on space-time and phase space. The combination of the star product formalism with geometric algebra can be seen as a program for a formal unification of physics. The consequences of this program on space time and phase space will be discussed in forthcoming papers. Especially it will be shown how constraints fit into this context.

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